ITP-UU-09/11
SPIN-09/11

INTRODUCTION TO THE THEORY OF BLACK HOLES*

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Version June 9, 2009

*Lectures presented at Utrecht University, 2009.

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1. Introduction

According to Newton's theory of gravity, the escape velocity v from a distance r from the center of gravity of a heavy object with mass m, is described by

$$\tfrac{1}{2}v^2 = \frac{G\,m}{r}\ . \tag{1.1}$$

What happens if a body with a large mass m is compressed so much that the escape velocity from its surface would exceed that of light, or, $v > c$? Are there bodies with a mass m and radius R such that

$$\frac{2G\,m}{R\,c^2} \geq 1\ ? \tag{1.2}$$

This question was asked as early as 1783 by John Mitchell. The situation was investigated further by Pierre Simon de Laplace in 1796. Do rays of light fall back towards the surface of such an object? One would expect that even light cannot escape to infinity. Later, it was suspected that, due to the wave nature of light, it might be able to escape anyway.

Now, we know that such simple considerations are misleading. To understand what happens with such extremely heavy objects, one has to consider Einstein's theory of relativity, both Special Relativity and General Relativity, the theory that describes the gravitational field when velocities are generated comparable to that of light.

Soon after Albert Einstein formulated this beautiful theory, it was realized that his equations have solutions in closed form. One naturally first tries to find solutions with maximal symmetry, being the radially symmetric case. Much later, also more general solutions, having less symmetry, were discovered. These solutions, however, showed some features that, at first, were difficult to comprehend. There appeared to be singularities that could not possibly be accepted as physical realities, until it was realized that at least some of these singularities were due only to appearances. Upon closer examination, it was discovered what their true physical nature is. It turned out that, at least in principle, a space traveller could go all the way in such a "thing" but never return. Indeed, also light would not emerge out of the central region of these solutions. It was John Archibald Wheeler who dubbed these strange objects "black holes".

Einstein was not pleased. Like many at first, he believed that these peculiar features were due to bad, or at least incomplete, physical understanding. Surely, he thought, those crazy black holes would go away. Today, however, his equations are much better understood. We not only accept the existence of black holes, we also understand how they can actually form under various circumstances. Theory allows us to calculate the behavior of material particles, fields or other substances near or inside a black hole. What is more, astronomers have now identified numerous objects in the heavens that completely match the detailed descriptions theoreticians have derived. These objects cannot be interpreted as anything else but black holes. The "astronomical black holes" exhibit no clash whatsoever with other physical laws. Indeed, they have become rich sources of knowledge about physical phenomena under extreme conditions. General Relativity itself can also now be examined up to great accuracies.

Astronomers found that black holes can only form from normal stellar objects if these represent a minimal amount of mass, being several times the mass of the Sun. For low mass black holes, no credible formation process is known, and indeed no indications have been found that black holes much lighter than this "Chandrasekhar limit" exist anywhere in the Universe.

Does this mean that much lighter black holes cannot exist? It is here that one could wonder about all those fundamental assumptions that underly the theory of quantum mechanics, which is the basic framework on which all atomic and sub-atomic processes known appear to be based. Quantum mechanics relies on the assumption that *every physically allowed configuration* must be included as taking part in a quantum process. Failure to take these into account would necessarily lead to inconsistent results. Mini black holes are certainly physically allowed, even if we do not know how they can be formed in practice. They can be formed in principle. Therefore, theoretical physicists have sought for ways to describe these, and in particular they attempted to include them in the general picture of the quantum mechanical interactions that occur in the sub-atomic world.

This turned out not to be easy at all. A remarkable piece of insight was obtained by Stephen Hawking, who did an elementary mental exercise: how should one describe relativistic quantized fields in the vicinity of a black hole? His conclusion was astonishing. He found that the distinction between particles and antiparticles goes awry. Different observers will observe particles in different ways. The only way one could reconcile this with common sense was to accept the conclusion that black holes actually do emit particles, as soon as their Compton wavelengths approach the dimensions of the black hole itself. This so-called "Hawking radiation" would be a property that all black holes have in common, though for the astronomical black holes it would be far too weak to be observed directly. The radiation is purely thermal. The Hawking temperature of a black hole is such that the Wien wave length corresponds to the radius of the black hole itself.

We assume basic knowledge of *Special* Relativity, assuming $c = 1$ for our unit system nearly everywhere, and in particular in the last parts of these notes also Quantum Mechanics and a basic understanding at an elementary level of Relativistic Quantum Field Theory are assumed. It was my intention not to assume that students have detailed knowledge of General Relativity, and most of these lectures should be understandable without knowing too much General Relativity. However, when it comes to discussing curved coordinates, Section 3, I do need all basic ingredients of that theory, so it is strongly advised to familiarize oneself with its basic concepts. The student is advised to consult my lecture notes "Introduction to General Relativity", http://www.phys.uu.nl/ thooft/lectures/genrel.pdf whenever something appears to become incomprehensible. Of course, there are numerous other texts on General Relativity; note that there are all sorts of variations in notation used.

2. The Metric of Space and Time

Points in three-dimensional space are denoted by a triplet of coordinates, $\vec{x} = (x, y, z)$, which we write as (x^1, x^2, x^3), and the time at which an event takes place is indicated by a fourth coordinate $t = x^0/c$, where c is the speed of light. The theory of Special Relativity is based on the assumption that all laws of Nature are invariant under a special set of transformations of space and time:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a^0_{\ 0} & a^0_{\ 1} & a^0_{\ 2} & a^0_{\ 3} \\ a^1_{\ 0} & a^1_{\ 1} & a^1_{\ 2} & a^1_{\ 3} \\ a^2_{\ 0} & a^2_{\ 1} & a^2_{\ 2} & a^2_{\ 3} \\ a^3_{\ 0} & a^3_{\ 1} & a^3_{\ 2} & a^3_{\ 3} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} ,$$

$$\text{or} \qquad x^{\mu\,\prime} = \sum_{\nu=0,\cdots,3} a^\mu_{\ \nu}\, x^\nu \ , \qquad \text{or} \qquad x' = A\, x \ , \tag{2.1}$$

provided that the matrix A is such that a special quantity remains invariant:

$$-c^2 t'^2 + x'^2 + y'^2 + z'^2 = -c^2 t^2 + x^2 + y^2 + z^2 \ ; \tag{2.2}$$

which we also write as:

$$\sum_{\mu,\nu=0,\cdots,3} g_{\mu\nu}\, x^\mu x^\nu \ \text{ is invariant,} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.3}$$

A matrix A with this property is called a Lorentz transformation. The invariance is Lorentz invariance. Usually, we also demand that

$$a^0_{\ 0} > 0 \ , \qquad \det(A) = +1 \ , \tag{2.4}$$

in which case we speak of *special* Lorentz transformations. The special Lorentz transformations form a group called $SO(3,1)$.

In what follows, *summation convention* will be used: in every term of an equation where an index such as the index ν in Eqs. (2.1) and (2.3) occurs exactly once as a superscript and once as a subscript, this index will be summed over the values $0, \cdots, 3$, so that the summation sign, $\sum$, does not have to be indicated explicitly anymore: $x^{\mu\,\prime} = a^\mu_{\ \nu} x^\nu$ and $s^2 = g_{\mu\nu}\, x^\mu x^\nu$. In the latter expression, summation convention has been implied twice.

More general linear transformations will turn out to be useful as well, but then (2.2) will not be invariant. In that case, we simply have to replace $g_{\mu\nu}$ by an other quantity, as follows:

$$g'_{\mu\nu} = (A^{-1})^\alpha_{\ \mu}\, (A^{-1})^\beta_{\ \nu}\, g_{\alpha\beta} \ , \tag{2.5}$$

so that the expression

$$s^2 = g_{\mu\nu}\, x^\mu x^\nu = g'_{\mu\nu}\, x^{\mu\,\prime} x^{\nu\,\prime} \tag{2.6}$$

remains obviously valid. Thus, Nature is invariant under general linear transformations provided that we use the transformation rule (2.5) for the tensor $g_{\mu\nu}$. This tensor will then be more general than (2.3). It is called the *metric tensor*. The quantity s defined by Eq. (2.6) is assumed to be positive (when the vector is spacelike), i times a positive number (when the vector is timelike), or zero (when x^μ is lightlike). It is then called the *invariant length* of a Lorentz vector x^μ .

In the general coordinate frame, one has to distinguish *co*-vectors x_μ from *contra*-vectors x^μ . they are related by

$$x_\mu = g_{\mu\nu} x^\nu \ ; \qquad x^\mu = g^{\mu\nu} x_\nu \ , \tag{2.7}$$

where $g^{\mu\nu}$ is the *inverse* of the metric tensor matrix $g_{\mu\nu}$. Usually, they are denoted by the same symbol; in a vector or tensor, replacing a subscript index by a superscript index means that, tacitly, it is multiplied by the metric tensor or its inverse, as in Eqs. (2.7).

3. Curved coordinates

The coordinates used in the previous section are such that they can be used directly to measure, or define, distances and time spans. We will call them Cartesian coordinates. Now consider just *any* coordinate frame, that is, the original coordinates (t, x, y, z) are completely arbitrary, in general mutually independent, differentiable functions of four quantities $u = \{u^\mu,\ \mu = 0, \cdots, 3\}$. Being differentiable here means that every point is surrounded by a small region where these functions are to a good approximation linear. There, the formalism described in the previous section applies. More precisely, at a given point x in space and time, consider points $x + \mathrm{d}x$, separated from x by only an infinitesimal distance $\mathrm{d}x$. Then we define $\mathrm{d}s$ by

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = g'_{\mu\nu}(u)\, \mathrm{d}u^\mu\, \mathrm{d}u^\nu \ . \tag{3.1}$$

The prime was written to remind us that $g_{\mu\nu}$ in the u coordinates is a different function than in the x coordinates, but in later sections this will be obvious and we omit the prime. Under a coordinate transformation, $g_{\mu\nu}$ transforms as Eq. (2.5), but now these coefficients are also coodinate dependent:

$$g'_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial u^\mu} \frac{\partial x^\beta}{\partial u^\nu}\, g_{\alpha\beta}(x) \ . \tag{3.2}$$

In the original, Cartesian coordinates, a particle on which no force acts, will go along a straight line, which we can describe as

$$\frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} = v^\mu = \ \text{constant}; \qquad v^\mu v^\mu = -1 \ ; \qquad \frac{\mathrm{d}^2 x^\mu(\tau)}{\mathrm{d}\tau^2} = 0 \ , \tag{3.3}$$

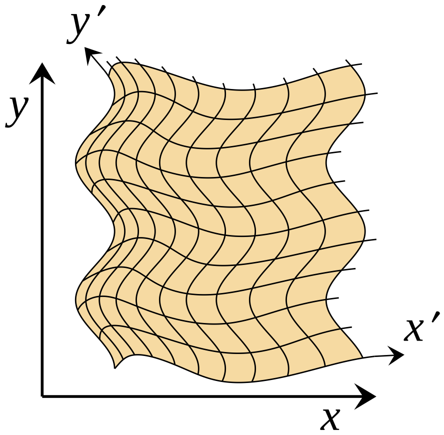


Figure 1: A transition from one coordinate frame $\{x, y\}$ to an other, curved coordinate frame $\{x', y'\}$.

where τ is the eigen time of the particle. In terms of *curved* coordinates $u^\mu(x)$, this no longer holds. Suppose that x^μ are arbitrary differentiable functions of coordinates u^λ. Then

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\partial x^\mu}{\partial u^\lambda}\frac{\mathrm{d}u^\lambda}{\mathrm{d}t} \; ; \qquad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \frac{\partial^2 x^\mu}{\partial u^\lambda \partial u^\kappa}\frac{\mathrm{d}u^\kappa}{\mathrm{d}\tau}\frac{\mathrm{d}u^\lambda}{\mathrm{d}\tau} + \frac{\partial x^\mu}{\partial u^\lambda}\frac{\mathrm{d}^2 u^\lambda}{\mathrm{d}\tau^2} \; . \tag{3.4}$$

Therefore, eq. (3.3) is then replaced by an equation of the form

$$g'_{\mu\nu}(u)\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} = -1 \; ; \qquad \frac{\mathrm{d}^2 u^\mu(\tau)}{\mathrm{d}\tau^2} + \Gamma^\mu_{\kappa\lambda}(u)\frac{\mathrm{d}u^\kappa}{\mathrm{d}\tau}\frac{\mathrm{d}u^\lambda}{\mathrm{d}\tau} = 0 \; , \tag{3.5}$$

where the function $\Gamma^\mu_{\kappa\lambda}(u)$ is given by

$$\Gamma^\mu_{\kappa\lambda}(u) = \frac{\partial u^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial u^\kappa \, \partial u^\lambda} \; . \tag{3.6}$$

Here, it was used that partial derivatives are invertible:

$$\frac{\partial u^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial u_\kappa} = \delta^\alpha_\kappa \; . \tag{3.7}$$

$\Gamma^\mu_{\kappa\lambda}$ is called the *connection* field. Note that it is symmetric under interchange of its two subscript indices:

$$\Gamma^\mu_{\kappa\lambda} = \Gamma^\mu_{\lambda\kappa} \; . \tag{3.8}$$

4. A short introduction to General Relativity

- A *scalar* function $\phi(x)$ of some arbitrary curved set of coordinates x^μ, ia a function that keeps the same values upon any coordinate transformation. Thus, a coordinate transformation $x^\mu \to u^\lambda$ implies that $\phi(x) = \phi'(u(x))$, where $\phi'(u)$ is the same scalar function, but written in terms of the new coordinates u^λ. Usually, we will omit the prime.

- A *co-vector* is any vectorial function $A_\alpha(x)$ of the curved coordinates x^μ that, upon a curved coordinate transformation, transforms just as the gradient of a scalar function $\phi(x)$. Thus, upon a coordinate transformation, this vectorial function transforms as

$$A_\alpha(x) = \frac{\partial u^\lambda}{\partial x^\alpha} A_\lambda(u) \ . \tag{4.1}$$

.

- A *contra-vector* $B^\mu(x)$ transforms with the inverse of that matrix, or

$$B^\mu(x) = \frac{\partial x^\mu}{\partial u^\lambda} B^\lambda(u) \ . \tag{4.2}$$

This ensures that the product $A_\alpha(x)B^\alpha(x)$ transforms as a scalar:

$$A_\alpha(x)B^\alpha(x) = \frac{\partial u^\lambda}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial u^\kappa} A_\lambda(u)B^\kappa(u) = A_\lambda(u)B^\lambda(u) \ , \tag{4.3}$$

where Eq. (3.7) was used.

- A *tensor* $A_{\alpha_1\alpha_2\cdots}^{\beta_1\beta_2\cdots}(x)$ is a function that transforms just as the product of covectors $A^1_{\alpha_1}, A^2_{\alpha_2}, \cdots$, and covectors $B_1^{\beta_1},\ B_2^{\beta_2},\ \cdots$. Superscript indices always refer to the contravector transformation rule and subscript indices to the covector transformation rule.

The *gradient* of a vector or tensor, in general, does not transform as a vector or tensor. To obtain a quantity that does transform as a true tensor, one must replace the gradient $\partial/\partial x^\mu$ by the so-called the *covariant derivative* D_μ, which for covectors is defined as

$$D_\mu A_\lambda(x) = \frac{\partial A_\lambda(x)}{\partial x^\mu} - \Gamma^\nu_{\mu\lambda}(x)A_\nu(x) \ , \tag{4.4}$$

for contravectors:

$$D_\mu B^\kappa(x) = \frac{\partial B^\kappa(x)}{\partial x^\mu} + \Gamma^\kappa_{\mu\nu}(x)B^\nu(x) \ , \tag{4.5}$$

and for tensors:

$$\begin{aligned} D_\mu A_{\alpha_1\alpha_2\cdots}^{\beta_1\beta_2\cdots}(x) = \frac{\partial}{\partial x^\mu} A_{\alpha_1\alpha_2\cdots}^{\beta_1\beta_2\cdots}(x) \ &- \ \Gamma^\nu_{\mu\alpha_1}(x) A_{\nu\alpha_2\cdots}^{\beta_1\beta_2\cdots}(x) - \Gamma^\nu_{\mu\alpha_2}(x) A_{\alpha_1\nu\cdots}^{\beta_1\beta_2\cdots}(x) - \cdots \\ &+ \ \Gamma^{\beta_1}_{\mu\nu}(x) A_{\alpha_1\alpha_2\cdots}^{\nu\beta_2\cdots}(x) + \cdots \ . \end{aligned} \tag{4.6}$$

In these expressions, $\Gamma^\nu_{\mu\lambda}$ is the connection field that we introduced in Eq. (3.6); there, however, we assumed a flat coordinate frame to exist. Now, this might not be so. In that case, we use the metric tensor $g_{\mu\nu}(x)$ to define $\Gamma^\nu_{\mu\lambda}$. It goes as follows. *If* we had a flat coordinate frame, the metric tensor $g_{\mu\nu}$ would be constant, so that its gradient vanishes. Suppose that we *demand* the covariant derivative of $g_{\mu\nu}$ to vanish as well. We have

$$D_\mu g_{\alpha\beta} = \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \Gamma^\nu_{\mu\alpha} g_{\nu\beta} - \Gamma^\nu_{\mu\beta} g_{\alpha\nu} \ . \tag{4.7}$$

Lowering indices using the metric tensor, this can be written as

$$D_\mu g_{\alpha\beta} = \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \Gamma_{\beta\mu\alpha} - \Gamma_{\alpha\mu\beta} \ . \tag{4.8}$$

Taking his covariant derivative to vanish, and using the fact that Γ is symmetric in its last two indices, we derive

$$\Gamma^\mu_{\kappa\lambda} = \tfrac{1}{2} g^{\mu\alpha} (\partial_\kappa g_{\alpha\lambda} + \partial_\lambda g_{\alpha\kappa} - \partial_\alpha g_{\kappa\lambda}) \ , \tag{4.9}$$

where $g^{\mu\alpha}$ is the inverse of $g_{\mu\nu}$, that is, $g_{\nu\mu}\, g^{\mu\alpha} = \delta^\alpha_\nu$, and ∂_κ stands short for the partial derivative: $\partial_\kappa = \partial/\partial u^\kappa$.

Eq. (4.9) will now be used as a *definition* of the connection field Γ. Note that it is always *symmetric* in its two subscript indices:

$$\Gamma^\mu_{\kappa\lambda} = \Gamma^\nu_{\lambda\kappa} \ . \tag{4.10}$$

This definition implies that $D_\mu g_{\alpha\beta} = 0$ automatically, as an easy calculation shows, and that the covariant derivatives of all vectors and tensors again transform as vectors and tensors.

It is important to note that the connection field $\Gamma^\alpha_{\alpha\beta}$ itself does *not* transform as a tensor; indeed, it is designed to fix quantities that aren't tensors back into forms that are. However, there does exist a quantity that is constructed out of the connection field that does transform as a tensor. This is the so-called *Riemann curvature.* This object will be used to describe to what extent space-time deviates from being flat. It is a tensor with four indices, defined as follows:

$$R^\mu_{\kappa\alpha\beta} = \partial_\alpha \Gamma^\mu_{\kappa\beta} - \partial_\beta \Gamma^\mu_{\kappa\alpha} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\kappa\beta} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\kappa\alpha} \ ; \tag{4.11}$$

in the last two terms, the index σ is summed over, as dictated by the summation convention. In the lecture course on general Relativity, the following statement is derived:

> *If V is a simply connected region in space-time, then the Riemann curvature $R^\mu_{\kappa\alpha\beta} = 0$ everywhere in V, if and only if a flat coordinate frame exists in V, that is, a coordinate frame in terms of which $g_{\mu\nu}(x) = g^0_{\mu\nu}$ everywhere in V.*

The *Ricci curvature* is a two-index tensor defined by contracting the Riemann curvature:

$$R_{\kappa\alpha} = R^\mu_{\kappa\mu\alpha} \ . \tag{4.12}$$

The Ricci scalar R is defined by contracting this once again, but because there are only two subscript indices, this contraction must go with the inverse metric tensor:

$$R = g^{\mu\nu} R_{\mu\nu} \ . \tag{4.13}$$

With some effort, one can derive that the Riemann tensor obeys the following (partial) differential equations, called *Bianchi identity*:

$$D_\alpha R^\mu_{\kappa\beta\gamma} + D_\beta R^\mu_{\kappa\gamma\alpha} + D_\gamma R^\mu_{\kappa\alpha\beta} = 0 \ . \tag{4.14}$$

From that, we derive that the Ricci tensor obeys

$$g^{\mu\nu} D_\mu R_{\nu\alpha} - \tfrac{1}{2} D_\alpha R = 0 \ . \tag{4.15}$$

5. Gravity

Consider a coordinate frame $\{x^\mu\}$ where $g_{\mu\nu}$ is time independent: $\partial_0 g_{\mu\nu} = 0$, and a particle that, at one instant, is at rest in this coordinate frame: $\mathrm{d}x^\mu/\mathrm{d}\tau = (1,0,0,0)$. Then, according to Eq. (3.5), it will undergo an acceleration

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{00} = \tfrac{1}{2} g^{ij} \partial_j g_{00} \ . \tag{5.1}$$

Since this acceleration is independent of the particle's mass, this is a perfect description of a gravitational force. In that case, $-\frac{1}{2}g_{00}$ can serve as an expression for the *gravitational potential* (note that, usually, g_{00} is negative). This is how the use of curved coordinates can serve as a description of gravity – in particular there must be curvature in the time dependence.

From here it is a small step to think of a space-time where the metric $g_{\mu\nu}(x)$ can be any differentiable function of the coordinates x. Coordinates x in terms of which $g_{\mu\nu}$ is completely constant do not have to exist. The gravitational field of the Earth, for instance, can be modelled by choosing $g_{00}(x)$ to take the shape of the Earth's gravitational potential. We then use Eqs. (3.5) and (4.9) to describe the motion of objects in free fall. This is the subject of the discipline called General Relativity.

Of course, no coordinate frame exists in which all objects on or near the Earth move in straight lines, and therefore we expect the Riemann curvature not to vanish. Indeed, we need to have equations that determine the connection field surrounding a heavy object like the Earth such that it describes the gravitational field correctly. In addition, we wish these equations to be invariant under Lorentz transformations. This is achieved if the equations can be written entirely in terms of vectors and tensors, *i.e.* all terms in the equations must transform as such under coordinate transformations. The gravitational equivalence principle requires that they transform as such under *all* (differentiable) curved coordinate transformations.

Clearly, the mass density, or equivalently, energy density $\varrho(\vec{x}, t)$ must play the role as a source. However, it is the 00 component of a tensor $T_{\mu\nu}(x)$, the mass-energy-momentum distribution of matter. So, this tensor must act as the source of the gravitational field.

Einstein managed to figure out the correct equations that determine how this matter distribution produces a gravitational field. $T_{\mu\nu}(x)$ is defined such that in flat space-time (with $c = 1$), $T^0_0 = -T_{00} = \varrho(x)$ is the energy distribution, $T_{i0} = T_{0i}$ is the *matter* *flow*, which equals the *momentum density*, and T_{ij} is the *tension*; for a gas or liquid with pressure p, the tension is $T_{ij} = -p\,\delta_{ij}$. The *continuity equation* in flat, local coordinates is

$$\partial_i T_{i\mu} - \partial_0 T_{0\mu} = 0 \ , \qquad \mu = 0,1,2,3. \tag{5.2}$$

Under general coordinate transformations, $T_{\mu\nu}$ transforms as a tensor, just as $g_{\mu\nu}$ does, see Eq. (3.2).

In curved coordinates, or in a gravitational field, the energy-momentum tensor does not obey the continuity equation (5.2), but instead:

$$g^{\mu\nu} D_\mu T_{\nu\alpha} = 0 \ , \tag{5.3}$$

So, the partial derivative ∂_μ has been replaced by the covariant derivative. This means that there is an extra term containing the connection field $\Gamma^\lambda_{\alpha\beta}$. This is the gravitational field, which adds or removes energy and momentum to matter.

Einstein's field equation now reads:

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = -8\pi G\, T_{\mu\nu} \ , \tag{5.4}$$

where G is Newton's constant. The second term in this equation is crucial. In his first attempts to write an equation, Einstein did not have this term, but then he hit upon inconsistencies: there were more equations than unknowns, and they were, in general, conflicting. Now we know the importance of the equation for energy-momentum conservation (5.3), written more compactly as $D_\mu T^\mu_\nu = 0$. It matches precisely the Bianchi identity (4.15) for the Ricci tensor, because that can also be written as

$$g^{\mu\nu} D_\mu (R_{\nu\alpha} - \tfrac{1}{2} R g_{\nu\alpha}) = 0 \ . \tag{5.5}$$

6. The Schwarzschild Solution

When Einstein found his equation, Eq. (5.4), end of 1915, he quickly derived approximate solutions, in order to see its consequences for observations, so that Eddington could set up his expedition to check the deflection of star light by the gravitational field of the sun. Einstein, however, did not expect that the equation could be solved exactly. It was Karl Schwarzschild, in 1916, who discovered that an exact, quite non-trivial solution can be found. We will here skip the details of its derivation, which is straightforward, though somewhat elaborate, and we will see more of that later. Schwarzschild's description of the metric $g_{\mu\nu}(x)$ that solves Einstein's equations is most easily expressed in the modern notation:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \;=\; -(1-2M/r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2 \mathrm{d}\Omega^2 \ , \tag{6.1}$$

$$\mathrm{d}\Omega^2 \;=\; \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2 \ , \tag{6.2}$$

where Newton's constant G has been absorbed in the definition of the mass parameter[1]: $M = Gm$. The advantage of this notation is that one can read off easily what the metric looks like if we make a coordinate transformation: just remember that $\mathrm{d}x^\mu$ is an infinitesimal displacement of a point in space and time. Notice from the dependence on $\mathrm{d}x^{0^2}$, that indeed, $-\frac{1}{2}g_{00}$ is the gravitational potential $-M/r$, apart from the constant 1.

Like other researchers in the early days, Schwarzschild himself was very puzzled by the singularity at $r = 2M$. He decided to replace the coordinate r by a "better" radial coordinate, let's call it $\tilde{r}$, defined as $\tilde{r} = (r^3 - (2M)^3)^{1/3}$. The reason for this substitution was that Schwarzschild used simplified equations that only hold if the space-time-volume element, $\det(g_{\mu\nu}) = -1$, and the shift he used simply subtracts an amount $(2M)^3$ from the space-time volume enclosed by r. Now, the singularity occurs at the "origin", $\tilde{r} = 0$. Schwarzschild died only months after his paper was published. His solution is now famous, but the substitution $r \to \tilde{r}$ (in the paper, the notation is different) was unnecessary. The apparent singularity at $r = 2M$ is easier to describe when it is kept right where it is, though indeed, we can use any coordinate frame we like to describe this metric. We emphasize that, whether or not the singularity is moved to the origin, only depends on the coordinate frame used, and has no physical significance whatsoever.[2]

One elegant coordinate substitution is the replacement of r and t by the *Kruskal-Szekeres coordinates* x and y, which are defined by the following two equations:

$$x\,y = \left(\frac{r}{2M} - 1\right) e^{r/(2M)} , \tag{6.3}$$

$$x/y = e^{t/(2M)} . \tag{6.4}$$

The angular coordinates θ and φ are kept the same. By taking the log of Eq. (6.3) and (6.4), and partially differentiating with respect to x and y, we read off:

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}r}{r - 2M} + \frac{\mathrm{d}r}{2M} = \frac{\mathrm{d}r}{2M(1 - 2M/r)} , \tag{6.5}$$

$$\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{2M} . \tag{6.6}$$

The Schwarzschild metric is now given by

$$\begin{aligned} \mathrm{d}s^2 &= 16M^2\left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{x\,y} + r^2\mathrm{d}\Omega^2 \\ &= \frac{32M^3}{r}\,e^{-r/(2M)}\mathrm{d}x\,\mathrm{d}y + r^2\mathrm{d}\Omega^2 . \end{aligned} \tag{6.7}$$

Notice now that, in the last expression, the zero and the pole at $r = 2M$ have cancelled out. The function $r(x, y)$ can be obtained by inverting the algebraic expression

[1] throughout these notes, we will denote the total mass of an object by m, and use the symbol M for Gm.

[2] There are some heated discussions of this on weblogs of amateur physicists who did not grab this point!

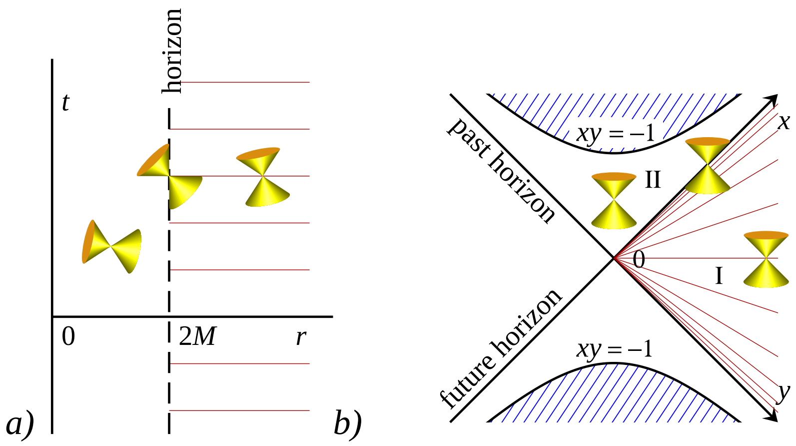


Figure 2: a) The black hole in the Schwarzschild coordinates r, t . The horizon is at $r = 2M$. b) Kruskal-Szekeres coordinates; here, the coordinates of the horizon are at $x = 0$ and at $y = 0$. The orientation of the local lightcones is indicated. Thin red lines are the time = Constant limes in the physical part of space-time.

(6.3) and is regular in the entire region $x\,y > -1$. In particular, nothing special seems to happen on the two lines $x = 0$ and $y = 0$. Apparently, there is no *physical singularity* or *curvature singularity* at $r \to 2M$. We do notice that the line $x = 0$, θ and φ both constant, is lightlike, since two neighboring points on that line obey $\mathrm{d}x = \mathrm{d}\theta = \mathrm{d}\varphi = 0$, and this implies that $\mathrm{d}s = 0$, regardless the value of $\mathrm{d}y$. similarly, the line $y = 0$ is lightlike. Indeed, we can also read off from the original expression (6.1) that if $r = 2M$, the lines with constant θ and φ are lightlike, as $\mathrm{d}s = 0$ regardless the value of $\mathrm{d}t$. The line $y = 0$ is called the *future horizon* and the line $x = 0$ is the *past horizon* (see Section 10).

An other important thing to observe is that Eq. (6.4) attaches a *real* value for the time t when x and y both have the same sign, such as is the case in the region marked I in Fig. $2\,b$, but if $x\,y < 0$, as in region II , the coordinate t gets an imaginary part. This means that region II is not part of our universe. Actually, t does not serve as a time coordinate there, but as a space coordinate, since there, $\mathrm{d}t^2$ enters with a positive sign in the metric (6.1). r is then the time coordinate.

Even if we restrict ourselves to the regions where t is real, we find that, in general, every point (r, t) in the physical region of space-time is mapped onto *two* points in the (x, y) plane: the points (x, y) and $(-x, -y)$ are mapped onto the same point (r, r) . This leads to the picture of a black hole being a *wormhole* connecting our universe to another universe, or perhaps another region of the space-time of our universe. However, there are no timelike or light like paths connecting these two universes. If this is a wormhole at all, it is a purely spacelike one.

7. The Chandrasekhar Limit

Consider Einstein's equation (5.4), and some spherically symmetric, stationary distribution of matter. Let $p(r)$ be the r dependent pressure, and $\varrho(r)$ the r dependent local mass density. An equation of state for the material relates p to ϱ. In terms of an auxiliary variable $m(r)$, roughly to be interpreted as the gravitational mass enclosed within a sphere with radius r, and putting $c = 1$, one can derive the following equations from General Relativity:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -G\frac{(\varrho + p)(m + 4\pi p\, r^3)}{r^2(1 - 2Gm/r)} , \tag{7.1}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\varrho\, r^2 . \tag{7.2}$$

These equations are known as the Tolman-Oppenheimer-Volkoff equations. The last of these, Eq. (7.2) seems to be easy to interpret. The first however, Eq. (7.1), seems to imply that not only energy but also pressure causes gravitational attraction. This is a peculiar consequence of the trace part of Einstein's equation (5.4). In many cases, such as the calculation of the gravitational forces between stars and planets, the pressure term cancels out precisely. This is when there is a boundary where the pressure vanishes.

The resulting space-time metric is calculated to take the form

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2) , \tag{7.3}$$

where

$$B(r) = \frac{1}{1 - 2Gm(r)/r} ; \tag{7.4}$$

and

$$\log(A(r)B(r)) = -8\pi G \int_r^\infty \frac{(p + \rho)\, r\, \mathrm{d}r}{1 - 2Gm(r)/r} . \tag{7.5}$$

Note that the Tolman-Oppenheimer-Volkoff equations (7.1) and (7.2) are exact, as soon as spherical symmetry and time independence are assumed.

There is no stable solution of the Tolman-Oppenheimer-Volkoff equations if, anywhere along the radius r, the enclosed gravitational mass $M(r) = Gm(r)$ exceeds the value $r/2$. If the equation of state allows the pressure to be high while the density ϱ is small, then large amounts of total mass will still show stable solutions, but if we have a liquid that is cool enough to show a fixed energy density ϱ^0 even when the pressure is low, the enclosed mass would be approximately $\frac{4}{3}\pi\varrho^0 r^3$, so with sufficiently large quantities of mass you can always exceed that limit. Thus, at sufficiently low temperatures, no stable, non-singular solution can exist if the baryonic mass N_B exceeds some critical value. Integrating inwards, one finds that there will be values of r where $M(r)/r$ exceeds the critical value $1/2$ so that $A(r)$ and $B(r)$ develop singularities.

Substituting some realistic equation of state at sufficiently low temperature, one derives that the smallest amount of total mass needed to make a black hole is then a little more than one solar mass. The *Chandrasekhar limit* refers to the *largest* amount of mass one can make of a substance where only electron pressure resists the gravitational attraction. This limit is about 1.44 solar masses.

One must ask what happens when larger quantities of mass are concentrated in a small enough volume. If no stable solution exists, this must mean that the system collapses under its own weight. What will happen to it?

8. Gravitational Collapse

An extreme case is matter of the form where the pressure p vanishes everywhere. This is called *dust*. When at rest, in a local Lorentz frame, dust has only an energy density $T_{00} = -\varrho$ while all other components of $T_{\mu\nu}$ vanish. In any other coordinate frame, the energy-momentum tensor takes the form

$$T^{\mu\nu}_{\text{dust}} = -\varrho(x) v^\mu v^\nu \ , \tag{8.1}$$

where v^μ is the local velocity $\mathrm{d}x^\mu/\mathrm{d}\tau$ of the dust grains.

In that case (and if we insist on spherical symmetry, so that the total angular momentum vanishes), gravitational implosion can never be avoided. It is instructive to show some simple exact solutions.

Consider as initial state a large sphere of matter contracting at a certain speed v. We could take v to be anything, but for simplicity we here choose it to be the velocity of light. Thus, at $t \to -\infty$ we take for the energy density $T^0_{\ 0}$ (and for simplicity $G = 1$),

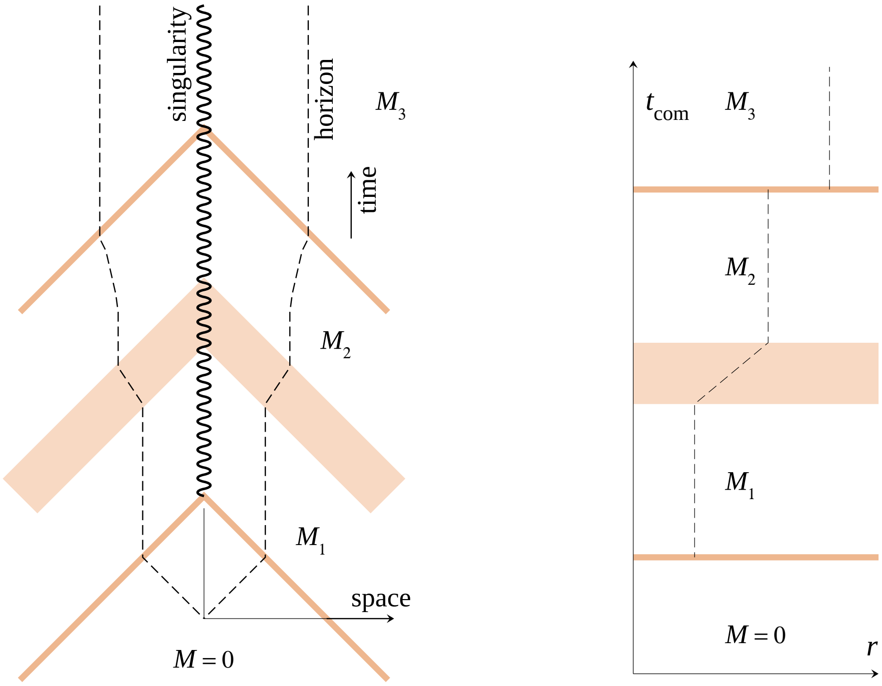
$$T^0_{\ 0}(r,t) \to \varrho_0\, \delta(r+t)/r^2 \ , \tag{8.2}$$

where the factor r^{-2} was inserted to ensure conservation of energy at infinity:

$$E = \int_0^\infty \mathrm{d}r\, 4\pi r^2 T^0_{\ 0}(r,t) = 4\pi\varrho_0 \ . \tag{8.3}$$

Thus, at $t \to -\infty$, matter is assumed to be confined into a thin, dense shell with radius $r \to |t|$.

With this initial condition, and the equation of state $p = 0$, it is not so difficult simply to guess the exact solution: we assume the metric to be stationary both before and after the passage of the dust shell, but while the dust shell passes there is a jump proportional to a theta step function. Both inside the dust shell and outside, spherical symmetry demands that the only admissible solution will then be the Schwarzschild metric with mass parameter M, however, M outside is different from M inside. With our initial condition (8.2), we have to choose M inside to be zero, but, for future use, we will also consider more general solutions, with different values for M_{in} and M_{out}. We will have to verify afterwards that the configuration obtained is indeed a correct solution, but we can already observe that spherical symmetry would not have left us any alternative. What has to be done now is to carefully formulate the *matching conditions* of the two regions at the location of the contracting dust shell.



a) b)

Figure 3: Several shells of matter (shaded lines) implode to form a black hole, whose mass M increases. In $a)$, time is neither the Schwarzschild time nor the $t_{\rm com}$ coordinate, but it indicates the causal order of events. The dotted line is the location of the horizon. In $b)$, the coordinates $t_{\rm com}$ and r are used. The dotted line here is the apparent Schwarzschild horizon $r = 2M$. Here, $0 < M_1 < M_2 < M_3$.

The contracting dust shell follows a lightlike geodesic in the radial direction, given by $\mathrm{d}s^2 = 0$, or

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\frac{A}{B}} = \frac{2M}{r} - 1 \,, \tag{8.4}$$

so that

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{-r}{r-2M} \quad \to \quad t(r) = -r - 2M\log(r-2M) \,. \tag{8.5}$$

This is a reason to use a modified coordinate frame both in the inside region and the outside region. Inside, we use the Schwarzschild metric with coordinates $(t_{\rm in},\, r,\, \theta,\, \varphi)$, and outside we use $(t_{\rm out},\, r,\, \theta,\, \varphi)$, but in both regions we make the transition to coordinates

$(t_{\rm com},\, r,\, \theta,\, \varphi)$, where

$$\begin{aligned} t_{\rm com} &= t_{\rm in} + r + 2M_{\rm in}\log(r - 2M_{\rm in}) \ , \\ t_{\rm com} &= t_{\rm out} + r + 2M_{\rm out}\log(r - 2M_{\rm out}) \ . \end{aligned} \tag{8.6}$$

Remember that for our original problem, $M_{\rm in} = 0$, so that, according to the initial condition (8.2), the dust shell moves at the orbit $t_{\rm com} = 0$. We call $t_{\rm com}$ the co-moving time. $t_{\rm com} = C^{\rm st}$ is a geodesic in both regions. The matching condition will now be that at the points $t_{\rm com} = 0$ the two regions are stitched together. The coordinates $r,\, \theta$ and φ will be the same for both regions (otherwise the metric $g_{\mu\nu}$ would show inadmissible discontinuities[3]). At $t_{\rm com} < 0$ we have $M = M_{\rm in}$ and at $t_{\rm com} > 0$ we have $M = M_{\rm out}$.

In terms of the new coordinates, the metric is

$$\begin{aligned} {\rm d}s^2 &= -A\left({\rm d}t - \sqrt{\frac{B}{A}}{\rm d}r\right)^2 + B{\rm d}r^2 + r^2{\rm d}\Omega^2 \\ &= -A{\rm d}t^2 + 2\,{\rm d}t\,{\rm d}r + r^2\,{\rm d}\Omega^2 \\ &= \left(-1 + \frac{\mu(t)}{r}\right){\rm d}t^2 + 2\,{\rm d}t\,{\rm d}r + r^2{\rm d}\Omega^2 \ , \end{aligned} \tag{8.7}$$

where

$$\mu(t) = 2M(t) \ , \qquad M(t) = \theta(t)M_{\rm out} + \theta(-t)M_{\rm in} \ , \tag{8.8}$$

and we dropped the subscript "com" for the time coordinate t.

In fact, any monotonously rising function $\mu(t)$ will be a solution to Einstein's equations where dust flows inwards with the speed of light. To check the solution now, let us evaluate the Ricci curvature for the metric (8.7) in these coordinates:

$$\begin{aligned} g_{00} = -1 + \frac{\mu}{r} \ , \quad g_{10} = g_{01} = 1 \ , \quad g_{11} = 0 \ , \\ g^{00} = 0 \ , \quad g^{10} = g^{01} = 1 \ , \quad g^{11} = 1 - \frac{\mu}{r} \ ; \end{aligned} \tag{8.9}$$

defining $\dot\mu = {\rm d}\mu/{\rm d}t$, we find[4] for the Christoffel symbols, $\Gamma_{\alpha\mu\nu} = \frac{1}{2}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_a g_{\mu\nu})$,

$$\begin{aligned} \Gamma_{000} = \frac{\dot\mu}{2r} \ , \quad \Gamma_{100} = \frac{\mu}{2r^2} \quad &, \quad \Gamma_{010} = \Gamma_{001} = -\frac{\mu}{2r^2} \ ; \\ \Gamma_{122} = -r \quad &, \quad \Gamma_{133} = -r\sin^2\theta \ , \\ \Gamma_{121} = \Gamma_{221} = r \quad &, \quad \Gamma_{313} = \Gamma_{331} = r\sin^2\theta \ , \\ \Gamma_{233} = -r^2\sin\theta\cos\theta \quad &, \quad \Gamma_{323} = \Gamma_{332} = r^2\sin\theta\cos\theta \ ; \end{aligned} \tag{8.10}$$

[3] In a space-time where the Ricci tensor is allowed to have a Dirac delta distribution, there must always be a coordinate frame such that: the *second* derivatives of the metric $g_{\mu\nu}$ may have delta peaks, but the *first* derivatives have at most discontinuities in the form of step functions, while the metric itself is continuous. If then a coordinate transformation is applied with a discontinuous first derivative, such as $x \to (a + b\,\theta(y))y$, with $a > 0$ and $a + b > 0$, the metric g_{xx} may show a discontinuity; compare the discontinuity in g_{00} as a function of $t_{\rm com}$ in Eq. (8.9).

[4] Remember that indices are raised and lowered by multiplying these fields with the metric tensor $g_{\mu\nu}$ or its inverse, $g^{\mu\nu}$.

$$\Gamma^0{}_{00}=\frac{\mu}{2r^2}\quad,\quad\Gamma^1{}_{10}=\Gamma^1{}_{01}=-\frac{\mu}{2r^2}\ ,$$
$$\Gamma^1{}_{00}=\frac{\dot\mu}{2r}+\frac{\mu}{2r^2}-\frac{\mu^2}{2r^3}\ ,$$
$$\Gamma^0{}_{22}=-r\ ,\quad\Gamma^0{}_{33}=-r\sin^2\theta\quad,\quad\Gamma^1{}_{22}=\mu-r\ ,\quad\Gamma^1{}_{33}=(\mu-r)\sin^2\theta\ ,$$
$$\Gamma^2{}_{12}=\Gamma^2{}_{21}=\frac{1}{r}\quad,\quad\Gamma^3{}_{13}=\Gamma^3{}_{31}=\cot\theta\ ;$$
$$\sqrt{-g}=r^2\sin\theta\quad,\quad\log\sqrt{-g}=2\log r+\log\theta\ . \tag{8.11}$$

Inserting these in the equation for the Ricci curvature,

$$R_{\mu\nu}=-(\log\sqrt{-g})_{,\mu,\nu}+\Gamma^\alpha_{\mu\nu,\alpha}-\Gamma^\beta_{\alpha\mu}\Gamma^\alpha_{\beta\nu}+\Gamma^\alpha_{\mu\nu}(\log\sqrt{-g})_{\alpha}\ , \tag{8.12}$$

we find that

$$R_{00}=\frac{\dot\mu}{r^2}\ , \tag{8.13}$$

while all other components of $R_{\mu\nu}$ in these coordinates vanish. It follows also for the trace[5] that $R=0$, because $g^{00}=0$. Hence

$$G\,T_{00}\ =\ \frac{-1}{8\pi}\frac{\dot\mu}{r^2}\ =\ \frac{-\dot M}{4\pi r^2}\ , \tag{8.14}$$

while all other components, notably also T_{11}, vanish. To see that this is indeed the energy momentum tensor of our dust shell, we note that, in our co-moving coordinates, the 4-velocity is $v^\mu=(0,-\Lambda,0,0)$, where Λ tends to infinity (our "dust" goes with the speed of light). From Eq. (8.9), we derive that $v_\mu=(-\Lambda,0,0,0)$, and we have agreement with Eq. (8.14) if

$$G\,\varrho=\frac{\dot M}{4\pi r^2\Lambda^2}\ . \tag{8.15}$$

We have

$$v_\mu=-\Lambda\partial_\mu t_{\,\rm com}\ , \tag{8.16}$$

so that, in the original Schwarzschild coordinates, where $t_{\rm com}$ is replaced by $t_{\rm in}$ or $t_{\rm out}$, according to Eq. (8.6),

$$v_\mu=\Lambda\left(-1,\ \frac{-1}{1-2M/r},\ 0,\ 0\right)\ ;\quad v^\mu=\Lambda\left(\frac{1}{1-2M/r},\ -1,\ 0,\ 0\right)\ ; \tag{8.17}$$

here M is the local mass parameter. From the second expression in Eq. (8.17), we see that indeed the shell is moving inwards, with the local speed of light. The situation is

[5]The fact that $R=0$ ensues from our choice to have the dust move with the speed of light. Of course, this is a limiting case, where all of the energy of the dust is kinetic, and the rest mass is negligible. The Ricci scalar R refers to this fact, that the dust has negligible rest mass.

sketched in Figures 3 a) and b) . One readily finds that, if $M_{\rm in}$ is taken to be zero, $M_{\rm out}$ is indeed the total energy E of our initial dust shell, defined while it was still at $r\rightarrow\infty$, in Eq. (8.3).

With a bit more work, this exercise can be repeated for dust going slower than the speed of light. What we see is that, just behind the first dust shell, the Schwarzschild metric emerges. Subsequent shells of dust go straight through the horizon, generating Schwarzschild metrics with larger mass parameters M . Taking M a continuous function of $t_{\rm com}$ leads to a description of less singular, spherically symmetric clouds of dust, coalescing to form a black hole.

At this stage it is very important to observe that this description of "gravitational collapse" allows for small perturbations to be added to it. For instance, one might assume a tiny amount of angular momentum or other violations of spherical symmetry in the initial state. This is what we mean when we say that the solution is 'robust'. The horizon at $r=2M$ might wobble a bit, but it cannot be removed by small perturbations only. This is because the horizon is not a true singularity but rather an artefact of the coordinates chosen. The singularity at $r=0$ on the other hand, is very sensitive to small perturbations, but it does not play a role in the physically observable properties of the black hole; it is well hidden way behind the horizon (an observation called *Cosmic Censorship*).

9. The Reissner-Nordström Solution

The Maxwell equations in curved space-time, when written in terms of the antisymmetric, covariant tensor $F_{\mu\nu}(x)$, are easy to find by replacing partial derivatives by covariant derivatives:

- The homogeneous Maxwell equation remains the same:

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0\ , \tag{9.1}$$

  because the contributions of the connection fields cancel out due to the complete antisymmetry under permutations of α , β , and γ . Hence, we still have a vector potential field A_μ obeying

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ . \tag{9.2}$$

- The inhomogeneous Maxwell equation is now

$$D_\mu F^\mu_\nu=g^{\alpha\beta}D_\alpha F_{\beta\nu}=-J_\nu\ , \tag{9.3}$$

  where $J_\nu(x)$ is the electro magnetic charge and current distribution. This can be rewritten as

$$\partial_\mu(\sqrt{-g}\,F^{\mu\nu})=-\sqrt{-g}\,J^\nu\ , \tag{9.4}$$

where the quantity g is the determinant of the metric:

$$g = \det_{\mu,\nu}(g_{\mu\nu}) \ , \tag{9.5}$$

so that we have the conservation law

$$\partial_\mu(\sqrt{-g}\, J^\mu) = 0 \ , \tag{9.6}$$

because the Maxwell field $F_{\mu\nu}$ is antisymmetric in its two indices.

- The energy momentum distribution of the Maxwell field is

$$T_{\mu\nu} = -F_{\mu\alpha}F_\nu{}^\alpha + (\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - J^\alpha A_\alpha)g_{\mu\nu} \ . \tag{9.7}$$

Spherical symmetry can still be used as a starting point for the construction of a solution of the combined Einstein-Maxwell equations for the fields surrounding a "planet" with electric charge Q and mass m. Just as Eq. (7.3) we choose

$$\mathrm{d}s^2 \ = \ -A\mathrm{d}t^2 + B\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,, \tag{9.8}$$

but now also a static electric field, defined by $E_i(x) = F_{0i} = -F_{i0}$:

$$E_r \ = \ E(r) \ ; \qquad E_\theta \ = \ E_\varphi \ = \ 0 \ ; \qquad \vec{B} \ = \ 0\,. \tag{9.9}$$

Let us assume that the source J^μ of this field is inside the planet and we are only interested in the solution outside the planet. So there we have

$$J^\mu \ = \ 0 \ . \tag{9.10}$$

Since $g \ = \ -ABr^4\sin^2\theta$, and $F^{0r} \ = \ -\frac{1}{AB}E(r)$, the inhomogeneous Maxwell law (9.4) implies

$$\partial_r\Big(\frac{E(r)r^2}{\sqrt{AB}}\Big) \ = \ 0 \ , \tag{9.11}$$

and consequently,

$$E(r) \ = \ \frac{Q\sqrt{AB}}{4\pi r^2} \ , \tag{9.12}$$

where Q is an integration constant, to be identified with electric charge since at $r\to\infty$ both A and B tend to 1.

The homogeneous parts of Maxwell's law are automatically obeyed because there is a field A_0 (potential field) with

$$E_r \ = \ -\partial_r A_0\,. \tag{9.13}$$

The field (9.12) contributes to $T_{\mu\nu}$:

$$T_{00} \;=\; -\,E^2/2B \;=\; -AQ^2/32\pi^2r^4 \;; \tag{9.14}$$

$$T_{11} \;=\; E^2/2A \;=\; BQ^2/32\pi^2r^4 \;; \tag{9.15}$$

$$T_{22} \;=\; -E^2r^2/2AB \;=\; -Q^2/32\pi^2r^2 \;, \tag{9.16}$$

$$T_{33} \;=\; T_{22}\sin^2\theta \;=\; -Q^2\sin^2\theta\,/32\pi^2r^2 \;. \tag{9.17}$$

We find

$$T^\mu_\mu \;=\; g^{\mu\nu}\,T_{\mu\nu} \;=\; 0 \;; \qquad R \;=\; 0\,, \tag{9.18}$$

a general property of the free Maxwell field. In this case we have (putting $G = 1$)

$$R_{\mu\nu} \;=\; -8\pi\,T_{\mu\nu}\,. \tag{9.19}$$

Herewith the Einstein equation (5.4) lead to the following solution:

$$A \;=\; 1 - \frac{2M}{r} + \frac{Q^2}{4\pi r^2} \;; \qquad B \;=\; 1/A\,. \tag{9.20}$$

This is the Reissner-Nordström solution (1916, 1918).

If we choose $Q^2/4\pi < M^2$ there are two "horizons", the roots of the equation $A = 0$:

$$r \;=\; r_\pm \;=\; M \pm \sqrt{M^2 - Q^2/4\pi}\,. \tag{9.21}$$

Again these singularities are artifacts of our coordinate choice and can be removed by generalizations of the Kruskal-Szekeres coordinates.

We have not shown the complete derivations of these solutions. In principle, the information given in these notes should suffice to derive them, but if further details are needed we refer to the various more elaborate texts in General Relativity. In these lecture notes we concentrate on the physical properties of the various metrics that were found.

10. Horizons

Consider the metric (6.1), and a light ray going radially inward. The equation for such a light ray is

$$\mathrm{d}s^2 = 0 \;; \quad \mathrm{d}\Omega = 0 \;, \qquad \text{or} \qquad \frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{1 - 2M/r} \;. \tag{10.1}$$

The solution of this equation is

$$t = t_0 \pm (r + 2M\log(r - 2M)) \;, \tag{10.2}$$

where t_0 is an integration constant, and we choose the minus sign, so that, at r very close to $2M$,

$$r(t) \to 2M + e^{(t_0 - t)/2M - 1} \;. \tag{10.3}$$

Similarly, a ray going radially outward is given by

$$r(t) \to 2M + e^{(t-t_1)/2M-1} . \tag{10.4}$$

Note that neither of these light rays ever pass through the barrier $r = 2M$. Rays going at an angle rather than radially also will not pass through this point. This is why this point is called a horizon. In general, a horizon forms when the coefficient A in the metric (9.8) tends to zero sufficiently fast.

In terms of the Kruskal coordinates, Eqs. (6.7), the inwards and outwards light rays are easier to find: the in-rays are at $x = C^{\underline{\mathrm{st}}}$, and the out-rays at $y = C^{\underline{\mathrm{st}}}$, where $C^{\underline{\mathrm{st}}}$ is any positive constant. In these coordinates, however, we see something new. The light rays do actually cross the horizon. Beyond the horizon, they hit upon the singularity at $xy = -1$, which is the point $r = 0$.

In solutions that are more general than the Schwarzschild solution, the horizon is defined to be the boundary line between two regions of space-time. Region I is the region defined by all space-time points x from which a geodesic can start, heading towards the future direction, that reaches the boundary at $r = \infty$. Region II is the collection of points that have no such geodesics attached to them. This means that particles, or indeed astronauts, cannot reach infinity from these points, regardless the value and direction of their initial velocity. The boundary line between I and II , the horizon, is a surface formed by lightlike geodesics pointing radially outwards.

There is also such a surface of lightlike geodesics pointing inwards. For the Schwarzschild solution, both horizons are at $r = 2GM$, but we see in the Kruskal frame that actually, the two horizons do not coincide.

In the Reissner-Nordström solution, we see that the function $A(r)$ has two zeros. The largest one, at

$$r = M + \sqrt{M^2 - Q^2/4\pi} , \tag{10.5}$$

coincides with the Schwarzschild horizon in the limit $Q \to 0$. It has the same properties as the Schwarzschild horizon.

The second horizon, at

$$r = M - \sqrt{M^2 - Q^2/4\pi} , \tag{10.6}$$

goes to the singularity $r = 0$ in the limit $Q \to 0$. But at finite Q it is also a lightlike surface.

Whatever happens within the horizon might be called physically irrelevant, since information concerning the interior region cannot be sent out using light rays. However, later we will see that quantum effects do depend on details of the horizon region, and to understand these, one may have to pass beyond the horizon.

The true, physical singularity that occurs at $r = 0$, is far hidden from observation; this singularity may be compared with singularities in physical equations when some parameters such as the time parameter become complex. Kepler's elliptical orbits, for instance, show delicate singularities at points in complex time, but the planets in the solar system do not seem to be bothered about that.

11. The Kerr and Kerr-Newman Solution

A fast rotating planet has a gravitational field that is no longer spherically symmetric but is only symmetric under rotations around the z-axis. We here just give the solution:

$$\begin{aligned} \mathrm{d}s^2 \;&=\; -\mathrm{d}t^2 + (r^2+a^2)\sin^2\theta \mathrm{d}\varphi^2 + \frac{2Mr(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi)^2}{r^2+a^2\cos^2\theta} \\ &\quad + (r^2+a^2\cos^2\theta)\Big(\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r^2-2Mr+a^2}\Big)\ . \end{aligned} \tag{11.1}$$

This solution was found by R. Kerr in 1963. To prove that this is indeed a solution of Einstein's equations requires patience but is not difficult. For a derivation using more elementary principles more powerful techniques and machinery of mathematical physics are needed. The free parameter a in this solution can be identified with angular momentum:

$$J = a\,M\ . \tag{11.2}$$

c) The Newman et al solution

For sake of completeness we also mention that rotating planets can also be electrically charged. The solution for that case was found by Newman et al in 1965. The metric is:

$$\mathrm{d}s^2 \;=\; -\frac{\Delta}{Y}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi)^2 + \frac{\sin^2\theta}{Y}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi)^2 + \frac{Y}{\Delta}\mathrm{d}r^2 + Y\mathrm{d}\theta^2\,, \tag{11.3}$$

where

$$Y \;=\; r^2 + a^2\cos^2\theta\ , \tag{11.4}$$

$$\Delta \;=\; r^2 - 2Mr + Q^2/4\pi + a^2\ . \tag{11.5}$$

The vector potential is

$$A_0 \;=\; -\frac{Qr}{4\pi Y}\ ; \qquad A_3 \;=\; \frac{Qra\sin^2\theta}{4\pi Y}\ . \tag{11.6}$$

Eq. (11.2) here also describes the total angular momentum in the solution. The Kerr-Newman solution is *the most general stationary solution for a black hole with electric charge*, if no matter is present.

As with the Reissner-Nordström solution, the zeros of the function $\Delta(r)$ ar not true singularities but rather coordinate singularities. The only genuine singularity in the curvature of space and time occurs where $Y(r,\theta)=0$, but this occurs only when both r and θ are zero: the singularity lies *along the equator at* $r=0$.

Exercise: show that when $Q=0$, Eqs. (11.1) and (11.3) coincide.

Exercise: find the non-rotating magnetic monopole solution by postulating a radial magnetic field.

Exercise: derive the gravitational field for a non-relativistic source by linearizing Einstein's equation (5.4), and use that to derive Eq. (11.2).

Exercise for the advanced student: describe geodesics in the Kerr solution.

12. Penrose diagrams

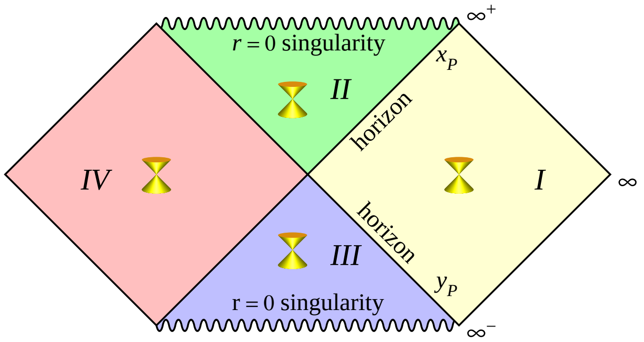


Figure 4: The Penrose diagram for the Schwarzschild solution

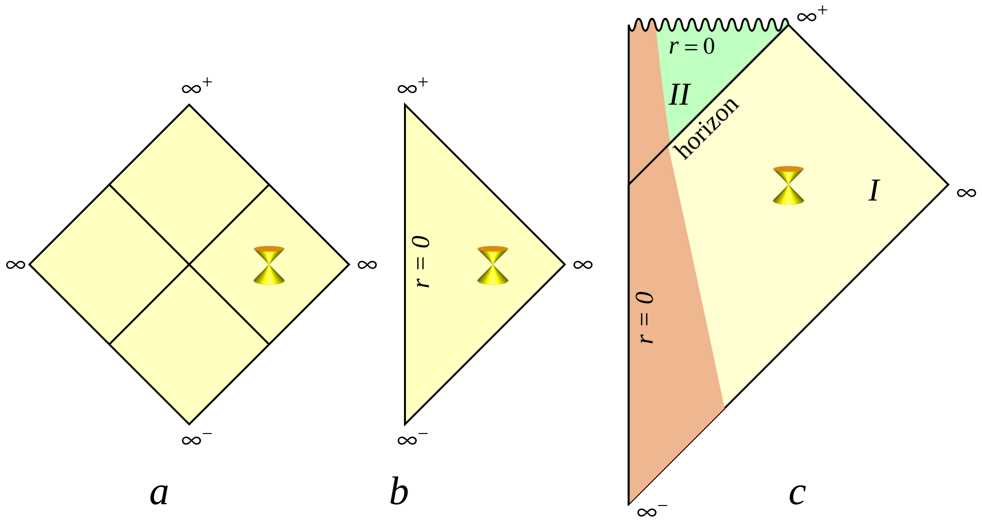


Figure 5: *a*) Penrose diagram for the Minkowski vacuum with Cartesian transverse coordinates, *b*) Minkowski space in polar coordinates; *c*) Penrose diagram for a black hole formed by matter (darker color represents matter falling in).

It is of interest to find coordinate systems that are such that they cover all of space-time that is continuously connected to the region that one has studied before, preferably avoiding any coordinate-induced singularities. This is not always possible, but we can try to choose the best possible coordinates. A good example is the Kruskal-Szekeres

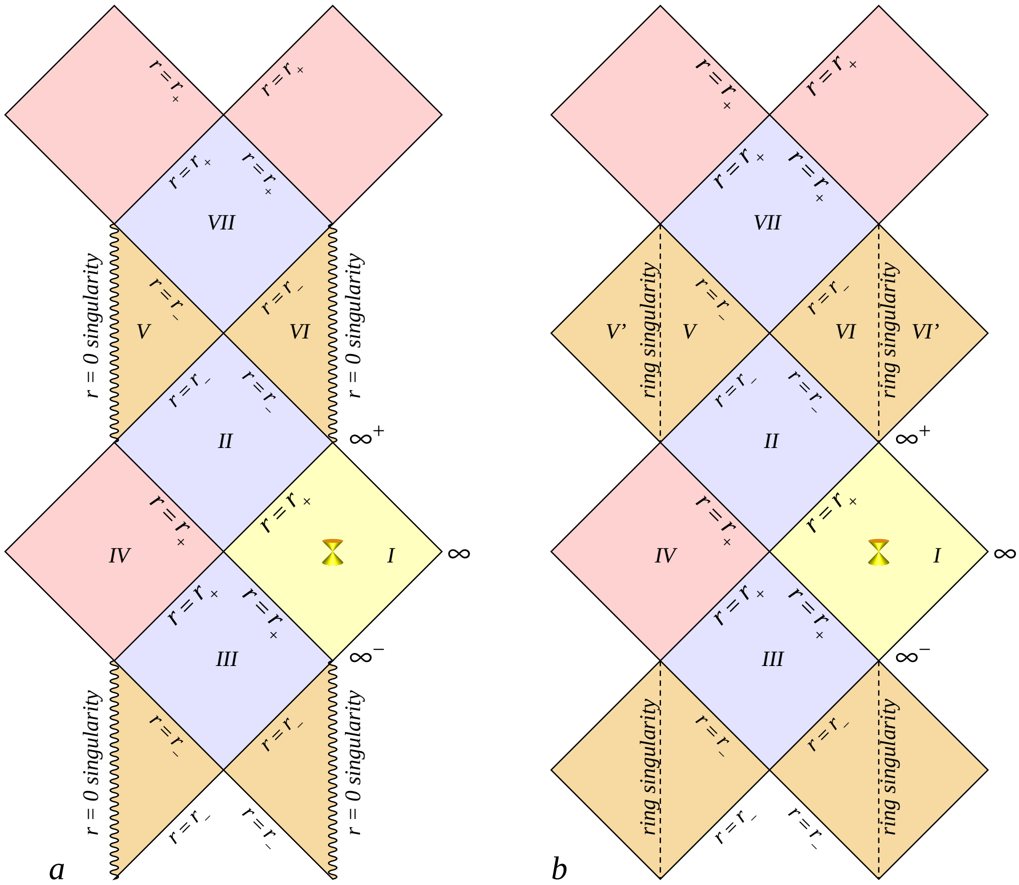


Figure 6: Penrose diagram for the Reissner-Nordström (a) and the Kerr black hole (b). The singularity at $r = 0$ is a natural boundary in case of Reissner-Nordström, but it is a *ring singularity* in case of Kerr- and Kerr-Newman, through which one can continue to asymptotically flat universes V' and VI', containing a negative mass Kerr Newman black hole.

coordinate system, (6.3) — (6.4), for the Schwarzschild black hole. At every point in the $x\,y$ frame of these coordinates, light rays are constrained to form angles of 45° or less with the vertical (vertical meaning the line $\mathrm{d}x + \mathrm{d}y = 0$). Or, light rays themselves form trajectories of the form $\mathrm{d}x \geq 0,\ \mathrm{d}y \leq 0$, where one of the equal signs is reached as soon as $\mathrm{d}\theta = \mathrm{d}\varphi = 0$.

Now these are not the only coordinates with this property. If the x coordinate is replaces by any monotonously increasing, differentiable function x_P of x , and y by any monotonously increasing differentiable function y_P of y , we still have the same property. This freedom we can use to obtain one other desirable feature: map the point $x = \infty$ to $x_P = 1$ and the same for y and y_P . Furthermore, we can assure that the $r = 0$ singularity is mapped onto a straight line, here the line $x_P - y_P = 1$. In the Schwarzschild case, this feature is reached if we choose

$$x = \tan(x_P\,\pi/2)\ , \qquad y = \tan(y_P\,\pi/2)\ . \tag{12.1}$$

Space-time is then sketched in Fig. 4.

A *Penrose diagram* now is a representation of two of the space-time coordinates in such a way that the local light cones always show that light rays go with a maximal velocity +1 to the right or -1 to the left, so that the fastest way to transmit information is by rays that are tilted by 45° to the left or to the right, such as is the case in Figure 4. The other two coordinates, θ and φ usually define a two-sphere. Characteristic boundaries are represented as much as possible by straight lines, which is usually possible and has the advantage that the entire space-time can be represented in a finite patch of the coordinates.

The diagram of Fig. 4 shows four regions of space-time, separated by horizons. Region I is the region that can be reached from infinity and from which one can also escape to infinity. Region II is the domain behind the horizon that can be reached by test objects falling in, but from where no escape back to infinity is possible. The $r = 0$ singularity lies in the future of any test object there. Region III is a domain that cannot be reached from infinity, but escape to infinity is allowed. Finally, region IV is only connected to the physical spacetime I by spacelike geodesics.

The Penrose diagram of flat Minkowski space-time is shown in Figure $5\,a$. Figure $5\,b$ describes a black hole formed by matter. We see that at negative times it corresponds to that of Minkowski space-time. At early times, the point $r = 0$ in 3-space forms a timelike geodesic; at later times it becomes spacelike

13. Trapped Surfaces

A black hole is characterized by the presence of a region in space-time from which no trajectories can be found that escape to infinity while keeping a velocity smaller than that of light. This implies the presence of *trapped surfaces* there. We start from the following definitions.

Consider a two-dimensional, closed, convex, spacelike surface S in a curved space-time. Let A be the surface area of S (calculated using the induced metric on S. Define a time coordinate t such that $t = 0$ on that surface. Suppose that our surface at $t = 0$ divides 3-space into two regions: an outer region V_1 and an inner region V_2. A small instant later, at time $t = \varepsilon$, 3-space is divided in three regions:

– an outer region V_1 that is spacelike separated from S,

– an inner region V_2 that is also spacelike separated from S, and

– a region V_3 between V_1 and V_2 that can be reached by timelike geodesics from S. Its boundary can be reached with lightlike geodesics from S.

Let S_1 be the boundary between V_1 and V_3 and S_2 be the boundary between V_2 and V_3. See Fig. 7. The surfaces S_1 and S_2 have areas A_1 and A_2. Now, we define the

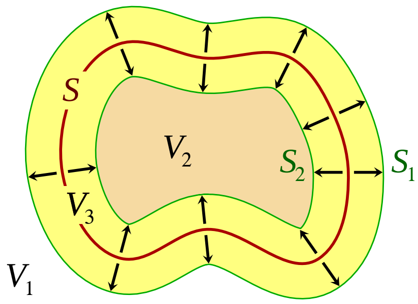


Figure 7: A surface S at $t = 0$ as described in the text. A little later, at $t = \varepsilon$, signals moving inwards and outwards divide 3-space into the regions V_1, V_2 and V_3.

expansion rates θ_1, θ_2 of these two surfaces as follows:

$$\theta_1 = \frac{\mathrm{d}A_1}{\mathrm{d}\varepsilon} , \qquad \theta_2 = \frac{\mathrm{d}A_2}{\mathrm{d}\varepsilon} . \tag{13.1}$$

Under non-exotic circumstances, such as in a flat space-time, certainly the outer surface expansion rate is positive: $\theta_1 > 0$. The inner one is usually negative. However, inside a black hole, we can have a trapped surface. S is called *trapped* iff both expansion rates are negative or zero:

$$\theta_1 \le 0 \quad \text{and} \quad \theta_2 \le 0 . \tag{13.2}$$

A surface is *marginally trapped* if the equal sign in Eq. (13.2) holds. For a pure Schwarzschild black hole, the surface $r = 2M$ is marginally trapped. This is because all light-like geodesics leaving this surface have $r = 2M$, so that its area, which in the local induced metric is $4\pi(2M)^2$, does not increase with time.

What happens in the presence of matter, when the solutions of Einstein's equations look a lot more complicated? In that case, we can still define trapped surfaces, and they obey a number of important theorems. One of the most important theorems is:

> If, in all locally regular coordinate frames, the matter distribution in a space-time obeys the constraint that the energy density is non-negative anywhere, or, in our notation,
>
> $$T_{00} \le 0 \tag{13.3}$$
>
> in all coordinate frames, then
>
> - a trapped surface stays trapped forever, and
> - the area of the largest trapped surface can only stay constant or increase.

The importance of this theorem is that it shows that black holes cannot disappear once they have been formed.[6] Indeed, other theorems show the inevitability of singularities

[6]Of course, we have not yet considered *quantum mechanical* effects. These can indeed cause black holes to shrink, and presumably disappear altogether, see Section 18.

forming inside trapped surfaces after some time, such as the $r = 0$ singularity of the Schwarzschild black hole.

We will not give the general proofs, but instead consider the much simpler spherically symmetric case. Consider the most general spherically symmetric metric,

$$\mathrm{d}s^2 = -A(r,t)\mathrm{d}t^2 + B(r,t)\mathrm{d}r^2 + C(r,t)\mathrm{d}r\mathrm{d}t + r^2\mathrm{d}\Omega^2 \ . \tag{13.4}$$

It is the direct generalization of the vacuum solution (6.1). Actually, the cross term $C(r,t)\mathrm{d}r\mathrm{d}t$ can easily be removed by a proper redefinition of the time coordinate t, so there is no loss of generality if we put

$$C(r,t) = 0 \tag{13.5}$$

(the r coordinate is fixed by demanding the angular dependence as in Eq. (13.4)).

It is now convenient to choose light cone coordinates x and y, which are defined by demanding that the lines $x =$constant and $y =$constant are in fact light rays $\mathrm{d}s = 0$. This implies that the coefficients for $\mathrm{d}x^2$ and $\mathrm{d}y^2$ must vanish. We then get the direct generalization of the Kruskal metric (6.7), which in the presence of matter reads

$$\mathrm{d}s^2 = 2A(x,y)\mathrm{d}x\mathrm{d}y + r^2(x,y)\mathrm{d}\Omega^2 \ . \tag{13.6}$$

From this metric, we calculate how it relates to the matter distribution $T_{\mu\nu}$. Here follows the calculation, which one might decide to skip at first reading, but we list it to enable the reader to check. The connection fields are easily calculated:

$$\begin{aligned}
&\Gamma^x_{xx} = \frac{\partial_x A}{A} \ , \quad \Gamma^x_{\theta\theta} = \frac{-r\,\partial_y r}{A} \quad , \quad \Gamma^x_{\varphi\varphi} = \frac{-r\,\partial_y r\sin^2\theta}{A} \ , \\
&\Gamma^y_{yy} = \frac{\partial_y A}{A} \ , \quad \Gamma^y_{\theta\theta} = \frac{-r\,\partial_x r}{A} \quad , \quad \Gamma^y_{\varphi\varphi} = \frac{-r\,\partial_x r\sin^2\theta}{A} \ , \\
&\Gamma^\theta_{x\theta} = \Gamma^\varphi_{x\varphi} = \frac{\partial_x r}{r} \quad , \quad \Gamma^\theta_{y\theta} = \Gamma^\varphi_{y\varphi} = \frac{\partial_y r}{r} \ , \\
&\Gamma^\theta_{\varphi\varphi} = -\cos\theta\,\sin\theta \quad , \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta \ ,
\end{aligned} \tag{13.7}$$

and all others are zero, except the ones obtained from the above by interchanging the two subscript indices.

From these, the Ricci tensor can be derived, and one obtains

$$\begin{aligned}
&R_{xx} = \frac{2\,\partial_x A\,\partial_x r}{Ar} - \frac{2\,\partial_x^2 r}{r} \quad , \qquad R_{yy} = \frac{2\,\partial_y A\,\partial_y r}{Ar} - \frac{2\,\partial_y^2 r}{r} \ , \\
&R_{xy} = \frac{\partial_x A\,\partial_y A}{A^2} - \frac{\partial_x\,\partial_y A}{A} - \frac{2\,\partial_x\partial_y r}{r} \ , \\
&R_{\theta\theta} = 1 - \frac{2\,\partial_x r\,\partial_y r + 2\,r\,\partial_x\partial_y r}{A} \quad , \qquad R_{\varphi\varphi} = \sin^2\theta\,R_{\theta\theta} \ .
\end{aligned} \tag{13.8}$$

This, we plug into Einstein's equation, Eq. (5.4), where we, temporarily, ignore the factor $8\pi G$:

$$T_{xx} = \frac{2\,\partial_x^2 r}{r} - \frac{2/,\partial_x A\,\partial_x r}{A\,r} = \frac{A}{r}\,\partial_x\left(\frac{\partial_x r}{A}\right) \quad , \qquad T_{yy} = \frac{A}{r}\,\partial_y\left(\frac{\partial_y r}{A}\right) \ , \tag{13.9}$$

$$T_{xy} = \frac{1}{r^2}\left(A - 2\,\partial_x r \partial_y r - 2\,r\,\partial_x\partial_y r\right) \quad = \quad \frac{A - \partial_x\partial_y(r^2)}{r^2}\ , \tag{13.10}$$

$$T_{\theta\theta} = -\frac{r\,\partial_x\partial_y \log A + 2\,\partial_x\partial_y r}{A} \qquad , \qquad T_{\varphi\varphi} = \sin^2\theta\ T_{\theta\theta}\ . \tag{13.11}$$

Now, keeping y zero or very small, we can regard the coordinate x as our time coordinate. In particular, we will use the first of Eq. (13.9). We claim that the positive energy condition will also require

$$T_{xx} \le 0\ . \tag{13.12}$$

Proof: Consider some given values for T_{xx}, T_{xy}, and T_{yy}. Now go to a coordinate frame $\{\tau,\ \varrho\}$, where τ serves as the new time variable, and

$$x = \lambda(\varrho + \tau)\ , \quad y = \frac{1}{\lambda}(\varrho - \tau)\ , \tag{13.13}$$

where the parameter λ is chosen sufficiently large. Since

$$\frac{\partial}{\partial\tau} = \lambda\frac{\partial}{\partial x} - \frac{1}{\lambda}\frac{\partial}{\partial y}\ , \tag{13.14}$$

we have

$$T_{\tau\tau} = \lambda^2 T_{xx} - 2T_{xy} + \lambda^{-2} T_{yy}\ . \tag{13.15}$$

Demanding this to be negative or zero for all λ implies Ineq. (13.12).

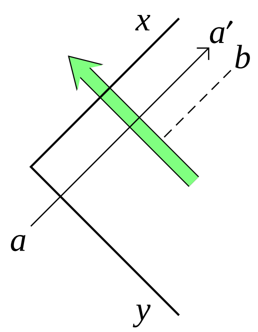


Figure 8: In the lightcone coordinates $\{x,\ y\}$, we may have a point $(x_0,\ y_0)$ where $\partial_x r = 0,\ \partial_y r > 0$, so that this point represents a trapped surface. If no matter is present, the line a, corresponding to $y = y_0$, then forms a series of trapped surfaces. But if matter falls in (green arrow), then beyond that point on the same line a', $\partial_x r < 0$, so that a new line b emerges at $y = y_1 > y_0$, where $\partial_x r = 0$. This is a new trapped surface with a larger area.

Now suppose that, at some positive value of A, we have a marginally trapped surface, so that, along a line $y = y_0 =$ constant, we have $\partial_x r = 0$. Since y runs in the negative time direction, we have $\partial_y r \ge 0$. If there is no matter around, then according to Eq. (13.9), r will keep the same value; the marginally trapped surface is then also a horizon. However, if there is matter around, obeying Ineq. (13.12), w the quantity $\partial_x r$ will be negative some time later. This means that the line $y = y_0$ is now well within this trapped surface. We can now go to a slightly larger value of y to find a marginally trapped surface. Since $\partial r/\partial y > 0$ (the surface is also trapped at the inner side), this surface has a larger r value, hence a larger area. Thus, as soon as matter falls in, the marginally trapped surface is replaced by a larger one. We can therefore conclude that the area of the horizon increases when matter falls in. See Fig. 8.

We used spherical symmetry for this simple argument, but it can be generalized to the non-symmetric case. All one needs to know is that all matter that falls through the horizon, has a positive energy density in any locally regular coordinate frame. In that case, the total area of the horizon can only increase.

14. The four laws of black hole dynamics

Consider the most general black hole solution, the Kerr-Newman solution (11.3). The horizon occurs where $\Delta = 0$, because at that point the lightlike geodesics going in and going out coincide:

$$\Delta(r) = 0 \ , \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = 0 \ , \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0 \ , \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{a}{r^2 + a^2} \ . \tag{14.1}$$

Defining the roots of Δ to be

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2/4\pi} \ , \tag{14.2}$$

we find that the horizon is at $r = r_+$, and its area is

$$\Sigma = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \ \sin\theta \, (r_+^2 + a^2) = 4\pi(r_+^2 + a^2) \tag{14.3}$$

(where the dependence of the function $Y(r, \theta)$ dropped out).

The free parameters of this solution are M, Q and a. For reasons to become clear shortly, we now wish to express these in terms of the three mutually independent parameters Q, J, and Σ. We have the following equations,

$$r_+^2 + a^2 = \Sigma/4\pi \ , \qquad r_+ r_- - a^2 = Q^2/4\pi \ , \qquad 2aM = a(r_+ + r_-) = 2J \ . \tag{14.4}$$

They allow us to eliminate r_+, r_- and a the following way:

$$\begin{aligned} 4J^2\left(a^2 - \frac{\Sigma}{4\pi}\right) + a^2\left(\frac{Q^2 + \Sigma}{4\pi}\right)^2 &= 0 \ ; \\ a^2 &= \frac{J^2\Sigma}{4\pi J^2 + \frac{1}{16\pi}(Q^2 + \Sigma)^2} \ . \end{aligned} \tag{14.5}$$

This gives us the dependence of M , the total mass/energy, on the independent parameters $Q, /J$ and Σ:

$$M^2(Q,\ J,\ \Sigma)\ =\ \frac{1}{16\pi}\Big(\frac{Q^4}{\Sigma}+2Q^2+\Sigma\Big)+\frac{4\pi J^2}{\Sigma}\ . \tag{14.6}$$

How does this change upon small variations of our free parameters? We write

$$\mathrm{d}M=\tau\,\mathrm{d}\Sigma+\Omega\,\mathrm{d}J+\phi\,\mathrm{d}Q\ . \tag{14.7}$$

These derivative functions are now derived to be

$$\tau=\frac{r_+-r_-}{16\pi(r_+^2+a^2)}\ ,\qquad \Omega=\frac{a}{r_+^2+a^2}\ ,\qquad \phi=\frac{Q\,r_+}{4\pi(r_+^2+a^2)}\ . \tag{14.8}$$

these now have the following interpretation. Ω is an angular velocity. For all systems with angular momentum J , the *increase in energy upon an increase of* J is the angular velocity. Indeed, this is the angular velocity that any object acquires when it goes through the horizon, see Eq. (14.1).

Similarly, ϕ is the *electrostatic potential* for a test charge crossing at the horizon. This is seen as follows. The vector potential (11.6) holds in the coordinates (t, r, θ, φ). If we want the vector potential for a test particle that rotates with angular velocity $\Omega = a/(r_+^2 + a^2)$, we have to transform to the co-rotating coordinates $(t, r, \theta, \tilde{\varphi})$, with $\tilde{\varphi} = \varphi - \Omega t$. The vector field transforms as follows:

$$\tilde{A}_\mu\mathrm{d}\tilde{x}^\mu=A_\mu\mathrm{d}x^\mu\ , \tag{14.9}$$

and since $\mathrm{d}\tilde{\varphi} = \mathrm{d}\varphi - \Omega\mathrm{d}t$, we have at $r = r_+$, where $Y = r_+^2 + a^2\cos^2\theta$,

$$\begin{aligned} A_0\mathrm{d}t+A_3\mathrm{d}\varphi\ &=\ \tilde{A}_0\mathrm{d}t+\tilde{A}_3(\mathrm{d}\varphi-\Omega\mathrm{d}t)\ \rightarrow\\ \tilde{A}_3=A_3\ ;\qquad \tilde{A}_0\ &=\ A_0+\Omega A_3\ =\ -\frac{Qr_+}{4\pi Y}\Big(1-\frac{a^2\sin^2\theta}{r_+^2+a^2}\Big)\ =\\ &=\ \frac{-Qr_+}{4\pi(r_+^2+a^2)}\ . \end{aligned} \tag{14.10}$$

This is the vector potential felt by the test charge with angular velocity Ω.

Σ is the area of the horizon, but what is τ? In all respects, this equation resembles the entropy equation in statistical mechanics. It was found by Bardeen, Carter and Hawking, and they noticed this similarity. In that case, τ acts as a temperature. It could not be the real temperature of a black hole, as was thought at first, because the black hole temperature is zero: nothing can come out, so also no thermal radiation. But the similarity with the entropy law went further. Due to the trapped surface theorems, we also know that the area of a horizon cannot decrease. This it has in common with entropy. thus, the second law of thermodynamics has an analogy in black holes:

the *second law of black hole physics* states that the total area of all horizons cannot decrease, just like the total entropy in thermodynamics.

The *first law of black hole physics* is equation (14.7). It states that the increase of mass of a black hole is the sum of all kinds of energy that is added to it. The amount $\mathrm{d}U = \tau \mathrm{d}\Sigma$ is then interpreted as heat energy.

The quantity

$$\tau = \frac{\kappa}{8\pi} = \frac{\sqrt{M^2 - a^2 - Q^2/4\pi}}{8\pi(r_+^2 + a^2)} \ , \tag{14.11}$$

cannot normally go to zero, and it takes the same value all across the horizon, just like the temperature for an object in equilibrium. κ is sometimes referred to as the "surface gravity" at the horizon. Very near the horizon, we again replace the angular coordinate φ by $\tilde\varphi = \varphi - \frac{a}{r_+^2 + a^2} t$, so that, at constant θ and $\tilde\varphi$, the metric (11.3) approaches

$$\mathrm{d}s^2 \rightarrow Y \left(\frac{-(r - r_+)(r_+ - r_-)}{(r_+^2 + a^2)^2} \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(r - r_+)(r_+ - r_-)} \right) \ . \tag{14.12}$$

The ratio between the time component and the space component is therefore the square of

$$\frac{(r - r_+)(r - r_-)}{r_+^2 + a^2} \ . \tag{14.13}$$

Differentiating with respect to r gives something that could be called the gravitational field at the horizon, which is κ in Eq. (14.11). We return to this topic in Section 18.

Thus, the *zeroth law* of black hole dynamics states that the "temperature" τ, or the "surface gravity", is constant on the horizon.

The *third law* would be that τ cannot be zero. This, indeed, is a delicate limit. It occurs when r_+ and r_- coincide, or

$$a^2 + Q^2/4\pi \rightarrow M^2 \ . \tag{14.14}$$

This is called the *extreme limit* of the Kerr-Newman black hole. It is dubious whether this limit can be reached in practice, but this has not been proven. The extreme limit of black holes has many special properties, and plays an important role in string theories.

15. Rindler space-time

Consider ordinary Minkowski space-time, described by the coordinates (t, x, y, z), where the metric is defined as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ . \tag{15.1}$$

It is instructive to transform towards the curved coordinates $(\tau,\ \varrho,\ \tilde x)$, where

$$\varrho\, e^{\tau} = z + t \ , \qquad \varrho\, e^{-\tau} = z - t \ , \qquad \tilde x = (x, y) \ . \tag{15.2}$$

A substitution

$$\tau \to \tau + \lambda \ , \tag{15.3}$$

where λ is a constant, would leave $z^2 - t^2$ invariant, and hence corresponds to a Lorentz transformation in Minkowski space-time. We can be sure that Nature's laws will not change, and so, in this curved coordinate frame, the laws of nature are invariant under translations of the new time variable τ. The metric in the new coordinates is

$$\mathrm{d}s^2 = -\varrho^2 \mathrm{d}\tau^2 + \mathrm{d}\varrho^2 + \mathrm{d}\tilde{x}^2 \ . \tag{15.4}$$

In this coordinate frame, therefore, an observer experiences a gravitational potential proportional to ϱ. Actually, at any position $\varrho = \varrho_0$, this observer would be tempted to redefine time as $\tilde{t} = \tau/\varrho_0$, so that the gravitational potential would feel as $V = \varrho/\varrho_0$, with gradient $1/\varrho_0$. Therefore, the actual gravitational field strength felt by the observer is inversely proportional to the distance from the origin.

This space-time is called *Rindler space-time*, and it is very instructive for the study of gravitational fields, since all physical phenomena observed in this world can be derived from what they are in Minkowski space-time without any gravitational field.

In fact, any small region very close to the horizon of a non-extremal black hole can de compared with Rindler space-time. Near $\theta \approx \pi/2$, replace the Schwarzschild coordinates as follows,

$$t/4M = \tau \ , \quad 8M(r-2M) = \varrho^2 \ , \qquad \tilde{x} = (2M\theta,\, 2M\varphi) \ , \tag{15.5}$$

then, close to $r \approx 2M$, the metric (6.1) is

$$\begin{aligned} \mathrm{d}s^2 \ &\approx \ -\frac{r-2M}{2M}\mathrm{d}t^2 + \frac{2M \ \mathrm{d}r^2}{r-2M} + \mathrm{d}\tilde{x}^2 \ = \\ &= \ -\varrho^2 \mathrm{d}\tau^2 + \mathrm{d}\varrho^2 + \mathrm{d}\tilde{x}^2 \ . \end{aligned} \tag{15.6}$$

In fact, the flat Minkowski coordinates z and t, (15.2), are closely related to the Kruskal-Szekeres coordinates (6.3)—(6.6).

Thus, we can find out about quantum phenomena near a horizon by studying them first in flat Minkowski space-time, then in Rindler space-time, and then in the black hole.

16. Euclidean gravity

Mathematical functions in space-time coordinates can often be extended to complex values of these coordinates. There, they continue to obey the same equations. In particular, it seems to be interesting to replace the time coordinate t by an imaginary time: $t = i\tilde{t}$. In Euclidean space, the metric then becomes

$$\mathrm{d}s^2 = +\mathrm{d}\tilde{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ . \tag{16.1}$$

The invariance group is then not the Poincaré group with the Lorentz group as its homogeneous part, but it now has the orthogonal group $SO(4)$. Thus, Lorentz transformations are replaced by ordinary rotations:

$$\begin{aligned} z' &= z \cos\gamma + \tilde t \sin\gamma \ , \\ \tilde t' &= -z \sin\gamma + \tilde t \cos\gamma \ , \end{aligned} \tag{16.2}$$

under which the metric (16.1) is invariant.

In Rindler spacetime, one can also extend to imaginary values of the Rindler time τ :

$$\begin{aligned} \tau &= i\tilde\tau \ , \qquad & z &= \varrho\cos\tilde\tau \ , \\ t &= i\tilde t \ , \qquad & \tilde t &= \varrho\sin\tilde\tau \ . \end{aligned} \tag{16.3}$$

This means that, in Euclidean space, the transition towards Rindler spacetime is nothing more than a transition to cylindrical coordinates. Rindler time translations are simply rotations in Euclidean space.

In the absence of matter, Einstein's equations in ordinary spacetime remain unchanged when we go to Euclidean spacetime. So we can take Schwarzschild's solution and extend it to Euclidean times:

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{r}\Big)\mathrm{d}\tilde t^2 + \frac{1}{1-2M/r}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ . \tag{16.4}$$

To see what happens at the horizon, we do the same substitution in the Kruskal-Szekeres coordinates. We find that Eqs. (6.3)—(6.7) turn into

$$|x| = |y| \ , \qquad x\, y \text{ is real} \ , \tag{16.5}$$

and, writing $|x| = |y| = \varrho$,

$$x = y^* = \varrho\, e^{i\tilde t/(4M)} = \xi + i\eta \ , \qquad \xi = \varrho\cos(\tfrac{\tilde t}{4M}) \ , \quad \eta = \varrho\sin(\tfrac{\tilde t}{4M}) \ . \tag{16.6}$$

$$\varrho^2 \ = \ \xi^2 + \eta^2 \ = \ \Big(\frac{r}{2M} - 1\Big)e^{r/(2M)} \ , \tag{16.7}$$

$$\mathrm{d}s^2 \ = \ \frac{32M^3}{r}e^{-r/(2M)}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + r^2\mathrm{d}\Omega^2 \ . \tag{16.8}$$

We see that the solution is rotationally invariant in (ξ, η) space, and the metric is regular at the origin. Time translations, also at very large values of r , are now rotations, and these are periodic. So, after a time translation over one period,

$$T = 8\pi M \ , \tag{16.9}$$

points in space and time return to their original positions. *At large distances, this spacetime is not excactly flat Euclidean spacetime, because points in Euclidean spacetime that are separated by one period T in Euclidean time have to be identified as being the same point.* If they were not the same point, singularities would arise at the origin of $(\xi,\ \eta)$ space. This observation will be very important later on (Section 18).

The region of Euclidean space-time that we described, where the metric is positive everywhere, only refers to the region $r \geq 2M$, so, one cannot go through the horizon here. The situation is sketched in Figure 9. We see that, asymptotically, this spacetime is not described by ordinary flat Euclidean spacetime, usually denoted as $\mathbb{R}_4$, but by a cylinder, $\mathbb{R}_3 \otimes S_1$, where S_1 stands for the circle.

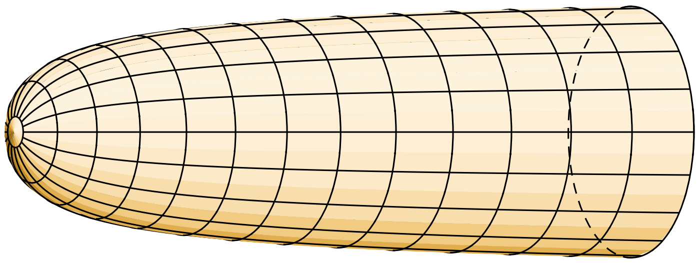


Figure 9: The Schwarzschild black hole in Euclidean gravity. Asymptotically, this spacetime is a cylinder.

17. The Unruh effect

For the following sections some basic knowledge of quantum field theory is required. We consider a quantized scalar field $\Phi(t,\,\vec x)$ in Minkowski space-time, and we shall investigate what it looks like in Rindler space-time. for the time being, we shall not need to include interactions, so we are talking of a free field. Its local commutation rules are

$$[\Phi(t,\vec x),\,\Phi(t,\vec x\,')]=0\ ,\qquad [\Phi(t,\vec x),\,\dot\Phi(t,\vec x\,')]=i\delta^3(\vec x-\vec x\,')\ . \tag{17.1}$$

Assuming this field to obey the Klein-Gordon equation,

$$(\vec\partial^{\,2}-\partial_0^2-m^2)\Phi=0\ , \tag{17.2}$$

one finds the Fourier mode expansion

$$\Phi(t,\vec x)\ =\ \int\frac{\mathrm{d}^3\vec k}{\sqrt{2k^0(\vec k)(2\pi)^3}}\Big(a(\vec k)e^{i\vec k\cdot\vec x-ik^0t}+a^\dagger(\vec k)e^{-i\vec k\cdot\vec x+ik^0t}\Big)\ , \tag{17.3}$$

$$\dot\Phi(t,\vec x)\ =\ \int\frac{-ik^0\mathrm{d}^3\vec k}{\sqrt{2k^0(\vec k)(2\pi)^3}}\Big(a(\vec k)e^{i\vec k\cdot\vec x-ik^0t}-a^\dagger(\vec k)e^{-i\vec k\cdot\vec x+ik^0t}\Big)\ , \tag{17.4}$$

where $k^0(\vec k)=\sqrt{\vec k^{\,2}+m^2}$ (always with the positive sign), and the operators $a(\vec k)$ and $a^\dagger(\vec k)$ obey the commutation rules for operators that respectively annihilate and create a particle:

$$[a(\vec k),\,a(\vec k\,')]=0\ ,\qquad [a(\vec k),\,a^\dagger(\vec k\,')]=\delta^3(\vec k-\vec k\,')\ , \tag{17.5}$$

In the Rindler space coordinates (15.2), the Klein-Gordon equation (17.2) reads

$$\Big((\varrho\partial_\varrho)^2-\partial_\tau^2+\varrho^2(\tilde\partial^{\,2}-m^2)\Big)\Phi=0\ . \tag{17.6}$$

Solutions periodic in τ are

$$\Phi_{\omega,\tilde k}(\tau,\varrho,\tilde x)\ =\ K\Big(\omega,\,\tfrac12\mu\varrho e^\tau,\,\tfrac12\mu\varrho e^{-\tau}\Big)e^{i\tilde k\cdot\tilde x}\ =\ K\Big(\omega,\,\tfrac12\mu\varrho,\,\tfrac12\mu\varrho\Big)e^{i\tilde k\cdot\tilde x-i\omega\tau}\ , \tag{17.7}$$

where $\mu^2=\tilde k^{\,2}+m^2$ and

$$K(\omega,\alpha,\beta)=\int_0^\infty\frac{\mathrm{d}s}{s}s^{i\omega}e^{-is\alpha+i\beta/s}\ . \tag{17.8}$$

We used the fact that the function K obeys

$$K(\omega,\,\sigma\alpha,\,\beta/\sigma)=\sigma^{-i\omega}K(\omega,\alpha,\beta)\ ; \tag{17.9}$$

it can be expressed in terms of the familiar Bessel and Hankel functions. Eq. (17.7) is readily obtained by taking one of the plane wave solutions in Minkowski space-time, $k^3=0$, $k^0=\mu$, rewriting it in terms of the Rindler coordinates ϱ and τ, and then Fourier transforming it with respect to τ. It is not difficult to verify directly (using partial integration in s) that the partial differential equation (17.6) is obeyed.

We now normalize the Fourier components of a field $\Phi(\tau,\varrho,\tilde x)$ with respect to τ as follows:

$$\begin{aligned}\Phi(\tau,\varrho,\tilde x)&=A(\tau,\varrho,\tilde x)+A^\dagger(\tau,\varrho,\tilde x)\ ,&(17.10)\\ A(\tau,\varrho,\tilde x)&=\int_{-\infty}^{\infty}\mathrm{d}\omega\int\frac{\mathrm{d}^2\tilde k}{\sqrt{2(2\pi)^4}}\,K(\omega,\,\tfrac12\mu\varrho,\,\tfrac12\mu\varrho)\,e^{i\tilde k\cdot\tilde x-i\omega\tau}a_2(\tilde k,\omega)\ ,&(17.11)\end{aligned}$$

so that the operator a_2 is identified as

$$a_2(\tilde k,\omega)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k^3}{\sqrt{2\pi k^0}}\,a(\vec k)\,e^{i\omega\ln\left(\frac{k^3+k^0}{\mu}\right)}\ , \tag{17.12}$$

where $k^0(\vec k)=\sqrt{k^{3\,2}+\mu^2}$. The inverse of this Fourier transform is

$$a(\vec k)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\sqrt{2\pi\,k^0}}\,a_2(\vec k,\omega)\,e^{-i\omega\ln\left(\frac{k^3+k^0}{\mu}\right)} \tag{17.13}$$

(remember that k^0 is a function of k^3, and $\partial k^0/\partial k^3=k^3/k^0$). Plugging Eq. (17.13) into Eq. (17.3) gives us Eq. (17.11), if the variable s in (17.8) is identified with $\frac{k^0-k^3}{\mu}=\frac{\mu}{k^0+k^3}$.

From the commutation rules (17.5), we derive similar commutation rules for a_2 :

$$[a_2(\tilde k,\omega),\,a_2^\dagger(\tilde k\,',\omega')]=\delta^2(\tilde k-\tilde k\,')\delta(\omega-\omega')\ . \tag{17.14}$$

However, before interpreting these as annihilation and creation operators, we must be aware of the fact that the integration in Eq. (17.11) also goes over negative values for ω. There, the operator $a(\tilde k,\omega)$ annihilates a *negative* amount of energy, so it really is a creation operator. Therefore, we must rearrange the positive and negative ω contributions when they are added in Eq. (17.10). To do this, it is convenient to note some properties of the functions K.

First, one has

$$K^*(\omega,\alpha,\beta)=K(-\omega,-\alpha,-\beta)\ . \tag{17.15}$$

Next, let $\alpha > 0$ and $\beta > 0$. In the definition (17.8), the integrand is bounded in the region $\mathrm{Im}(s) \le 0$. Therefore, one may rotate the integration contour as follows:

$$s \rightarrow s\,e^{-i\phi}\ , \qquad 0 \le \phi \le \pi\ . \tag{17.16}$$

Taking the case $\phi = \pi$, we find that

$$K(-\omega,\alpha,\beta) = \int_0^\infty \frac{\mathrm{d}s}{s} s^{-i\omega} e^{-\pi\omega} e^{is\alpha - i\beta/s} = e^{-\pi\omega} K^*(\omega,\alpha,\beta) \quad \text{if} \quad \alpha > 0\ ,\ \beta > 0\ , \tag{17.17}$$

and similarly one has

$$K(-\omega,\alpha\beta) = e^{+\pi\omega} K^*(\omega,\alpha,\beta) \quad \text{if} \quad \alpha < 0\ ,\ \beta < 0\ . \tag{17.18}$$

This allows us to collect the positive and negative ω contributions in Eqs. (17.10) and (17.11) as follows. In the region $\varrho > 0$, we have

$$\begin{aligned}\Phi(\tau,\varrho,\tilde{x}) \ &= \ \int_0^\infty \mathrm{d}\omega\, e^{-i\omega\tau} \int \frac{\mathrm{d}^2\tilde{k}\, e^{i\tilde{k}\cdot\tilde{x}}}{\sqrt{2(2\pi)^4}} K(\omega, \tfrac{1}{2}\mu\varrho, \tfrac{1}{2}\mu\varrho) \\ &\quad \times \Big(a_2(\tilde{k},\omega) + e^{-\pi\omega} a_2^\dagger(-\tilde{k},-\omega) \Big) + \text{H.c.}\end{aligned} \tag{17.19}$$

In the opposite quadrant of Rindler space, where $\varrho < 0$, we have

$$\begin{aligned}\Phi(\tau,\varrho,\tilde{x}) \ &= \ \int_0^\infty \mathrm{d}\omega\, e^{-i\omega\tau} \int \frac{\mathrm{d}^2\tilde{k}\, e^{i\tilde{k}\cdot\tilde{x}}}{\sqrt{2(2\pi)^4}} K(\omega, \tfrac{1}{2}\mu\varrho, \tfrac{1}{2}\mu\varrho) \\ &\quad \times \Big(a_2(\tilde{k},\omega) + e^{+\pi\omega} a_2^\dagger(-\tilde{k},-\omega) \Big) + \text{H.c.}\end{aligned} \tag{17.20}$$

At this point, it is opportune to define the new creation and annihilation operators a_I, $a_I^\dagger$, a_{II} and $a_{II}^\dagger$, applying the following Bogolyubov transformation, when $\omega > 0$:

$$\begin{pmatrix} a_I(\tilde{k},\omega) \\ a_I^\dagger(-\tilde{k},\omega) \\ a_{II}(\tilde{k},\omega) \\ a_{II}^\dagger(-\tilde{k},\omega) \end{pmatrix} = \frac{1}{\sqrt{1-e^{-2\pi\omega}}} \begin{pmatrix} 1 & 0 & 0 & e^{-\pi\omega} \\ 0 & 1 & e^{-\pi\omega} & 0 \\ 0 & e^{-\pi\omega} & 1 & 0 \\ e^{-\pi\omega} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2(\tilde{k},\omega) \\ a_2^\dagger(-\tilde{k},\omega) \\ a_2(\tilde{k},-\omega) \\ a_2^\dagger(-\tilde{k},-\omega) \end{pmatrix} \tag{17.21}$$

Inverting this, we see that we have

$$\begin{pmatrix} a_2(\tilde{k},\omega) \\ a_2^\dagger(-\tilde{k},\omega) \\ a_2(\tilde{k},-\omega) \\ a_2^\dagger(-\tilde{k},-\omega) \end{pmatrix} = \frac{1}{\sqrt{1-e^{-2\pi\omega}}} \begin{pmatrix} 1 & 0 & 0 & -e^{-\pi\omega} \\ 0 & 1 & -e^{-\pi\omega} & 0 \\ 0 & -e^{-\pi\omega} & 1 & 0 \\ -e^{-\pi\omega} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_I(\tilde{k},\omega) \\ a_I^\dagger(-\tilde{k},\omega) \\ a_{II}(\tilde{k},\omega) \\ a_{II}^\dagger(-\tilde{k},\omega) \end{pmatrix} \tag{17.22}$$

We then see that, at $\varrho > 0$, the field Φ depends only on a_I and $a_I^\dagger$, while at $\varrho < 0$, the field Φ depends only on a_{II} and $a_{II}^\dagger$. The normalization has again been chosen such that

$$\begin{aligned} [a_I(\tilde{k},\omega),\, a_I^\dagger(\tilde{k}\,',\omega')] \ = \ [a_{II}(\tilde{k},\omega),\, a_{II}^\dagger(\tilde{k}\,',\omega')] \ &= \ \delta(\omega-\omega')\delta^2(\tilde{k}-\tilde{k}\,')\ , \\ [a_I,\, a_{II}] \ = \ [a_I,\, a_{ii}^\dagger] \ &= \ 0\ . \end{aligned} \tag{17.23}$$

The importance of this is the following. The operators a_I and a_{II} are defined such that they only annihilate objects with positive energy, and their hermitean conjugates only create positive energies. In the quadrant $\varrho > 0$, *only* the combination created by $a_I^\dagger$ can be detected using the field $F(\tau, \varrho, \tilde{x})$, and in the quadrant $\varrho < 0$ only the other operators, a_{II} act. It is important to realize this, because if we had not paid attention to this, we could have kept the original operators, defining $a_2(\tilde{k}, \omega)$ and $a_2^\dagger(\tilde{k}, -\omega)$ as annihilation operators at $\omega > 0$, without the apparent need for a Bogolyubov transformation.

Just because the fields depend on time as $e^{-i\omega\tau}$, the Rindler space Hamiltonian, that is, the operator that generates a boost in the Rindler time parameter τ, is

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega \int \mathrm{d}^2\tilde{k}\, a_2^\dagger(\tilde{k}, \omega) a_2(\tilde{k}, \omega) \\ &= \int_0^\infty \mathrm{d}\omega\, \omega \int \mathrm{d}^2\tilde{k}\, \Big(a_I^\dagger(\tilde{k}, \omega) a_I(\tilde{k}, \omega) - a_{II}^\dagger(\tilde{k}, \omega) a_{II}(\tilde{k}, \omega) \Big) = H_R^I - H_R^{II} \end{aligned} \tag{17.24}$$

One may also verify that, if $\mathcal{H}_M(\vec{x})$ is the Hamiltonian *density* in Minkowski space-time at time $t = 0$, then

$$H_R^I = \int_{\varrho>0} \mathrm{d}^3\vec{x}\, \varrho\, \mathcal{H}_M(\vec{x}) \ , \qquad H_R^{II} = \int_{\varrho<0} \mathrm{d}\vec{x} |\varrho| \mathcal{H}_M(\vec{x}) \ . \tag{17.25}$$

Consequently, all observables constructed out of the fields ϕ in the quadrant I where $\varrho > 0$ commute with H_R^{II} and the observables in quadrant II, $\varrho < 0$ commute with H_R^I.

The *vacuum state* as experienced by an observer in Rindler space is the ground state of H_R^I, and since $[H_R^I,\, H_R^{II}] = 0$, we can also have the ground state of H_R^{II}. We refer to this state as $|0,\, 0\rangle_R$. It is called the *Boulware vacuum.*

However, the Boulware vacuum is not at all the lowest energy state in Minkowski space-time. Let is take that state, $|\Omega\rangle_M$, which is defined as

$$a(\vec{k})|\Omega\rangle_M = 0 \ , \qquad a_2(\tilde{k}, \omega)|\Omega\rangle_M = 0 \ . \tag{17.26}$$

This describes empty space-time as experienced by an observer who is stationary in Minkowski space, or freely falling in the gravitational field of Rindler space-time. What does $|\Omega\rangle_M$ look like to the observer who is at a fixed position $(\varrho,\, \tilde{x})$, with $\varrho > 0$, in Rindler space? For this observer, the operator a_I describes the particles. We have

$$\begin{aligned} a_I(\tilde{k}, \omega)|\Omega\rangle_M &= e^{-\pi\omega} a_{II}^\dagger(-\tilde{k}, \omega)|\Omega\rangle_M \ , \\ a_{II}(\tilde{k}, \omega)|\Omega\rangle_M &= e^{-\pi\omega} a_I^\dagger(-\tilde{k}, \omega)|\Omega\rangle_M \ . \end{aligned} \tag{17.27}$$

These equations are easy to solve. We find

$$|\Omega\rangle_M = \prod_{\tilde{k},\omega} \sqrt{1 - e^{-2\pi\omega}} \sum_{n=0}^{\infty} |n\rangle_I\, |n\rangle_{II}\, e^{-\pi\omega n} \ , \tag{17.28}$$

where the square root is added for normalization. Note that

$$H_R|\Omega\rangle_M = (H_R^I - H_R^{II})|\Omega\rangle_M = 0 \ , \tag{17.29}$$

which confirms that the Minkowski vacuum is Lorentz invariant; remember that H_R is the generator of Lorentz boosts.

Consider any observable $\mathcal{O}$ in the positive sector of Rindler space-time. It must commute with H_R^{II}, and therefore

$$\mathcal{O}(|\psi\rangle_I|\psi'\rangle_{II}) = |\lambda'\rangle_{II}(\mathcal{O}|\psi\rangle_I) \ . \tag{17.30}$$

Let us concentrate on only one sector of $\tilde{k}$ and ω. There, the expectation value of such an operator is

$$\begin{aligned} {}_M\langle\Omega|\mathcal{O}|\Omega\rangle_M &= (1-e^{-2\pi\omega})\sum_{n_1,n_2} {}_{II}\langle n_1|_I\langle n_1|\mathcal{O}|n_2\rangle_I|n_2\rangle_{II}e^{-\pi\omega(n_1+n_2)} \\ &= \sum_{n\ge 0} {}_I\langle n|\mathcal{O}|n\rangle_I\, e^{-2\pi n\omega}(1-e^{-2\pi\omega}) \ = \ \mathrm{Tr}\,(\mathcal{O}\,\varrho_\Omega) \ , \end{aligned} \tag{17.31}$$

where ϱ_Ω is the *density matrix* $C\,e^{-\beta H_I}$ corresponding to a thermal state ate the temperature

$$T = 1/(k\,\beta) = 1/(2\pi k) \ , \tag{17.32}$$

where k is Boltzmann's constant: $\beta = 1/(kT)$. Note that this temperature is dimensionless. This is because the time unit, τ, in the Rindler coordinates (15.2), is dimensionless. In Section 15, we saw that, at the distance $\varrho = \varrho_0$ from the Rindler horizon, the strength of the gravitational field is $\frac{1}{\varrho_0}$, and furthermore that time has to be rescaled by a factor ϱ_0. Therefore, we conclude that an observer who is being accelerated by a gravitational field with strength g in relativistic units, experiences radiation with a temperature $T = g/(2\pi k)$. This is the Unruh effect.

18. Hawking radiation

The region of space-time in the vicinity of the horizon of a black hole, approximately takes the form of Rindler space, that is, the Schwarzschild time coordinate t relates to Rindler time there as in Eq. (15.5): $t = 4M\tau$. Therefore, the temperature experienced there is given by

$$kT_H = 1/8\pi M = 1/8\pi G m_{\mathrm{BH}} \ . \tag{18.1}$$

This is the Hawking temperature of a black hole.

The value for this temperature could have been derived more intuitively as follows. The free energy F of any thermal quantum system is computed as

$$e^{-\beta F} = \mathrm{Tr}\,(e^{-\beta H}) \ , \tag{18.2}$$

where $\beta = 1/kT$, and k is the Boltzmann constant. From this expression for F , one can compute the thermal average of any operator $\mathcal{O}$ of a system. Assume a small disturbance:

$$e^{-\beta F(J)} = \mathrm{Tr}\,(e^{-\beta(H+J\mathcal{O})})\ , \tag{18.3}$$

then we have, keeping β fixed:

$$\begin{aligned}\langle\mathcal{O}\rangle_T \quad &= \quad \frac{\sum_E \langle E|\mathcal{O}|E\rangle e^{-\beta E}}{\sum_E e^{-\beta E}} = \frac{-\frac{\partial}{\beta\partial J}\mathrm{Tr}\, e^{-\beta H-\beta J\mathcal{O}}|_{J=0}}{\mathrm{Tr}\, e^{-\beta H}} = \\ &-\frac{\partial}{\beta\partial J}\log(\mathrm{Tr}\,(e^{-\beta H-\beta J\mathcal{O}}))|_{J=0} \ = \ \frac{\partial F}{\partial J}|_{J=0}\ .\end{aligned} \tag{18.4}$$

The operator $e^{-\beta H}$ happens to be the evolution operator e^{-iHP} for a time period $P = -i\beta$, and taking the trace means that the evolution operator is connected to itself after this period in imaginary time. So, this essentially means that quantum mechanics over a space-time that is periodic in imaginary time is equivalent to working out thermal expectation values of operators at a temperature T equal to

$$P = \beta\hbar = \hbar/(kT)\ , \tag{18.5}$$

where we re-inserted the constant $\hbar$.

The Unruh temperature $g/(2\pi k)$ is thus connected to the fact that it refers to a Rindler space that has periodicity $2\pi/g$, and the Hawking temperature $1/(8\pi kM)$ follows from the periodicity (16.9) derived for the Kruskal spacetime in Section 16.

We can now also understand why the temperature, or equivalently, the surface gravity (14.11), cannot depend on the position along the horizon: if a solution is periodic with period P at one spot, it cannot have any different periodicity elsewhere, since the space-time must still have the same analytic form after any number of periods in this particular Euclidean direction.

For the general Kerr-Newman solution, the metric near the horizon approaches Eq. (14.12). Writing

$$r - r^+ = \varrho^2\ , \qquad \mathrm{d}r = 2\varrho\,\mathrm{d}\varrho\ , \tag{18.6}$$

we find

$$\begin{aligned}\mathrm{d}s^2 \to \qquad & \frac{4Y}{r_+ - r_-}\Big(- \kappa^2\varrho^2\mathrm{d}\tau^2 + \mathrm{d}\varrho^2\Big)\ , \\ & \kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}\ .\end{aligned} \tag{18.7}$$

So, rather than the "surface gravity", we should view κ as the parameter that scales the time variable in Rindler space-time at the horizon. Therefore, the Hawking temperature of a Kerr Newman black hole is

$$kT = \kappa/2\pi = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{M^2 - a^2 - Q^2/4\pi}}{2M^2 - Q^2/4\pi + 2M\sqrt{M^2 - a^2 - Q^2/4\pi}}\ . \tag{18.8}$$

Clearly then, the parameter that looked like a temperature when we phrased the "four laws of black hole dynamics", really is a temperature! It is actually 4 times the parameter τ that was introduced in section 14. Scaling everything else there accordingly, we find that the actual entropy S, as it occurs in the equation $\mathrm{d}U = T\mathrm{d}S + \cdots$ is 4 times smaller:

> The temperature of a black hole equals $kT = 4\kappa$, where κ is the "surface gravity". The entropy of a black hole equals
>
> $$S = \tfrac{1}{4} k\Sigma \ , \tag{18.9}$$
>
> where Σ is the area of the horizon.

Note that we kept Boltzmann's constant k in our descriptions of temperature and entropy. Units for the temperature could be chosen such that it is one.

The fact that there are particles with a certain temperature near the horizon of a black hole, means that some of these thermally excited particles can escape to infinity, and be observed there. Indeed, with its temperature $T = 1/(8\pi kM)$, there will be radiation emerging from the horizon. The intensity of the radiation will be proportional with T^4 close to the horizon, and the total energy loss per unit of time due to this radiation will be approximately

$$U = C_1 \Sigma T^4 = C_2 M^2 M^{-4} = C_2 M^{-2} \ , \tag{18.10}$$

where C_1 and C_2 are constants depending not only on the geometric details of the black hole (what will be its apparent surface area as seen from infinity?), but also, weakly, on temperature, because the number of particle types participating in the radiation depends on whether the temperature is sufficiently high to excite particles with given rest masses.

Let us nevertheless take Eq. (18.10) as a rough approximation. Assuming conservation of total mass/energy (General Relativity would be inconsistent if we did not), we must conclude that, if left by itself, a black hole should loose mass:

$$\begin{aligned} &\frac{\mathrm{d}M}{\mathrm{d}t} \approx -C_2 M^{-2} \ \rightarrow \ \frac{\mathrm{d}M^3}{\mathrm{d}t} \approx -3C_2 \ ; \\ &M(t) \approx (3C_2)^{\frac{1}{3}} (t_0 - t)^{\frac{1}{3}} \ . \end{aligned} \tag{18.11}$$

Clearly, at some moment $t = t_0$ the black hole must disappear altogether. What exactly happens at that moment, however, cannot be understood without a more complete understanding of quantum gravity than we possess today. We do expect this to be quite a bang, because the total mass-energy emitted in the last second, turns out to be formidable, once we put our conventional units back in.

19. The implication of black holes for a quantum theory of gravity

In thermodynamics, the entropy S of a system with no other adjustable parameters obeys

$$T\mathrm{d}S = \mathrm{d}U \ , \tag{19.1}$$

where U is the energy stored as heat. The quantity F, defined as

$$F = U - TS \ , \tag{19.2}$$

is called the Helmholtz free energy, and it obeys

$$\mathrm{d}F = -S\mathrm{d}T \ , \qquad \text{or} \qquad S = -\frac{\partial F}{\partial T} \ . \tag{19.3}$$

In statistical physics, the free energy F has been identified by the equation

$$e^{-\beta F} = \sum_E e^{-\beta E} = \mathrm{Tr}\,(e^{-\beta H}) \ , \tag{19.4}$$

where $\beta = 1/(kT)$, while k is Boltzmann's constant and H is the quantum Hamiltonian. The sum is over all quantum states $|E\rangle$. The quantity $e^{-\beta E}$ is the Boltzmann factor describing the probability of any state $|E\rangle$ to occur when there is thermal equilibrium.

The total energy U is then given by

$$U \;=\; \frac{\sum_E E\, e^{-\beta E}}{\sum_E e^{-\beta E}} \;=\; \frac{\mathrm{Tr}\,(H\, e^{-\beta H})}{\mathrm{Tr}\,(e^{-\beta H})} \;=\; \frac{-\frac{\partial}{\partial\beta} e^{-\beta F(\beta)}}{e^{-\beta F}} \;=\; \frac{\partial}{\partial\beta}(\beta F) \ . \tag{19.5}$$

Eq. (19.2) can be written as

$$S = k\beta(U - F) \ , \tag{19.6}$$

and we derive

$$S/k \;=\; \frac{\mathrm{Tr}\,(\beta H e^{-\beta H})}{\mathrm{Tr}\,(e^{-\beta H})} + \log \mathrm{Tr}\,(e^{-\beta H}) \;=\; \log \mathrm{Tr}\,(e^{\langle \beta H\rangle - \beta H}) \ , \tag{19.7}$$

where U has been written as an average: $U = \langle H\rangle$.

In these expressions, it was assumed that our system is a grand canonical ensemble. We also can consider micro canonical ensembles, which may be a collection of many systems but always in such a way that the total energy U is kept fixed. In that case, the sum is only over all states with the same energy $E = U$. Then, the exponent in Eq. (19.7) is 1, and the entropy is then seen to be

$$S = k \log(\mathrm{Tr}\,(1)) \ , \tag{19.8}$$

or, the entropy is nothing but the logarithm of the total number of states over which we sum. This is a quite general result: *In a quantum system, the entropy is k times the logarithm of the total number of quantum states that can describe the system we are looking at.*

For a black hole, this is a fundamental feature. Since here, we conclude from Eq. (18.9) that the total number of "black hole microstates" is given by

$$\varrho = C \cdot e^{\Sigma/4} \ . \tag{19.9}$$

C is an unknown constant. This is because entropy is always defined apart from an unknown additive constant. In Eq. (19.9), this is a multiplicative constant, which is unknown.

Much research is going into identifying these quantum states. Can we write a Schrödinger equation for black holes? This question is compounded by the fact that the pure Minkowski vacuum state, when transformed into Rindler coordinates, emerges as a density matrix, $e^{-\beta H}$, which is a mixture of quantum states. Does a collapsing system smoothly transmute from a pure quantum state into a mixed state? This we do not believe. To understand this situation better, we must study the collapsing system in different coordinate frames, and include the consideration that the metric of space and time itself must be subject to quantum oscillations. This is beyond the scope of this lecture course.

The application of thermodynamics to black holes could be criticized for the following reason: let us try to calculate the 'specific heat' of a black hole. What is $\mathrm{d}U/\mathrm{d}T$, or, the amount of heat needed to raise the temperature by one degree? The temperature is given by Eq. (18.1), so

$$U = 1/(8\pi kT) \ . \tag{19.10}$$

Therefore,

$$\frac{\mathrm{d}U}{\mathrm{d}T} = -1/(8\pi kT^2) \ < \ 0 \ , \tag{19.11}$$

so the temperature goes down when heat is added. This means that the black hole is fundamentally unstable thermally.

But there is another way to derive that the number of black hole microstates is the exponent of $\frac{1}{4}\times$ the area Σ of the horizon. Consider a quantum mechanical description of the process of capturing something. The cross section for capture in a Schwarzschild black hole can roughly be estimated to be[7]

$$\sigma(\vec{k}) = \pi r_+^2 = 4\pi M^2 \ , \tag{19.12}$$

where $\vec{k}$ is the momentum of the ingoing particle. Now we also know the probability W for emitting a particle, which is given by the thermal probability:

$$W\mathrm{d}t = \frac{\sigma(\vec{k})v}{V} e^{-\beta_H E}\mathrm{d}t \ , \tag{19.13}$$

where β_H is the inverse Hawking temperature:

$$\beta_H = 1/kT_H = 8\pi M \ . \tag{19.14}$$

and V is the volume of the space where the particle is released.

[7]The actual value will be considerably larger, and momentum dependent, because the orbits cannot be straight lines, but in the present argument only the order of magnitude is of significance.

Now we *assume* that the process is also coverned by a Schrödinger equation. This means that there exist quantum mechanical transition amplitudes,

$$\mathcal{T}_{\text{in}} = {}_{\text{BH}}\langle M+GE|\,|M\rangle_{\text{BH}}|E\rangle_{\text{in}} \ , \tag{19.15}$$

$$\text{and} \qquad \mathcal{T}_{\text{out}} = {}_{\text{BH}}\langle M|\langle E|\,|M+GE\rangle_{\text{BH}} \ , \tag{19.16}$$

where the states $|M\rangle_{\text{BH}}$ represent black hole states with mass M/G, and the states $|E\rangle$ are states of surrounding particles with total energy E, confined to a volume V. In terms of these amplitudes, using the so-called *Fermi Golden Rule*, the cross section σ and the emission probabilities W can be written as

$$\sigma = |\mathcal{T}_{\text{in}}|^2 \varrho(M+GE)/v \ , \tag{19.17}$$

$$W = |\mathcal{T}_{\text{out}}|^2 \varrho(M)\frac{1}{V} \ , \tag{19.18}$$

where $\varrho(M)$ stands for the presumed density of quantum levels of a black hole with mass M. The factor v^{-1} in Eq. (19.17) is a kinematical factor, and the factor V^{-1} in Eq. (19.18) arises from the normalization of the wave functions.

Time reversal invariance would relate $\mathcal{T}_{\text{in}}$ to $\mathcal{T}_{\text{out}}$. To be precise, all we need is CPT invariance, since a parity transformation P and a charge conjugation C have no effect on our calculation of σ. Dividing the expressions (19.17) and (19.18), and using Eq, (19.13), one finds:

$$\frac{\varrho(M+GE)}{\varrho(M)} = e^{\beta E} = e^{8\pi M} \ . \tag{19.19}$$

This is easy to integrate:

$$\frac{\mathrm{d}\log\varrho(M)}{\mathrm{d}M} = 8\pi M/G \ , \tag{19.20}$$

$$\varrho(M) = C\,e^{4\pi M^2/G} = e^{S/k} \ . \tag{19.21}$$

Thus, we found a direct expression for the density of quantum levels, which was now *defined* to be the logarithm of an entropy. It coincides with the thermodynamic expression (19.9).

Clearly, this analysis suggests that black holes obey a Schrödinger equation describing the evolution of internal quantum states, and we can estimate rather precisely the dimensionality of this internal Hilbert space. It is as if there is one Boolean degree of freedom per unit of area A_0 of the horizon:

$$\varrho = 2^{A/A_0} \ , \qquad A_0 = 4G\log 2 \ . \tag{19.22}$$

But how can we understand the details of this Schrödinger equation? Curiously, the answer to this question does not appear to follow from any of the first principles that have been discussed so-far. To the contrary, there seems to be a contradiction. According to Hawking's derivation of the radiation process, any black hole, regardless its past, ends up as a *thermodynamically mixed state*. Would this also hold for a black hole that started out as a collapsing star *in a quantum mechanically pure state*? Can pure states evolve into mixed states? Not according to conventional quantum mechanics.

From a physical point of view, the distinction between pure states and mixed states for macroscopic objects is pointless. Black holes should be regarded as being macroscopic. So, it is very likely that what we perceive as a mixed state is actually a pure state whose details we were unable to resolve. However, if that is true, the derivation given by Hawking is wanting. We should search for a more precise analysis.

In fact, approximations and simplifications were made in Hawking's derivation. In particular, in and outgoing particles were assumed not to interact with one another. Usually, this is a reasonable assumption. However, in this case, it is easy to observe that particles from the collapse entering the black hole at early times and Hawking particles leaving the black hole at late times, meet each other very close to the horizon. The center-of-mass energy that this encounter represents diverges exponentially with the time lapse, so it can easily surpass the mass-energy of the entire universe. Such passings cannot go without mutual interactions; they would merge to form gigantic black holes, but long before that happens, our analysis has become invalid. This is where our procedures should be repaired. This, however, is difficult and research is in progress.

In the mean time, there have been other developments, notably in string theory. According to string theory, D-brane configurations form soliton-like configurations that play the role of black holes. For these black holes, the microstates can be counted, provided that they are not too far separated from the extreme limit. The counting appears to confirm the result (19.21).

In a unified theory of all particles and forces, the primary building blocks are the heaviest and most compact forms of matter. We see that such forms of matter are black holes. There cannot be "other" primary forms of matter, since all massive objects must be surrounded by gravitational fields, *i.e.*, they are black holes. The properties of these black holes, in turn, must be determined by field theories describing particles at their horizons. So, the question of unifying all forces and matter forms ends up in a logical spiral. This makes the problem interesting and challenging from a theoretical point of view. A more precise and coherent theoretical approach might lead to further insights.

20. The Aechelburg-Sexl metric

We wish to find the space-time metric surrounding a particle that goes almost with the speed of light towards the positive z-direction. Consider the schwarzschild metric in the case of a very light mass m:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{2\mu}{r}(\mathrm{d}t^2 + \mathrm{d}r^2) \ , \tag{20.1}$$

where $\mu = Gm$, and $\mathrm{d}x^2$ is the flat metric $\mathrm{d}\tilde{x}^{\,2} - \mathrm{d}t^2$. This, we rewrite as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{2\mu}{r}(u\cdot \mathrm{d}x)^2 + \frac{2\mu}{r}\mathrm{d}r^2 \ , \qquad r = \sqrt{x^2 + (u\cdot x)^2} \ , \tag{20.2}$$

where

$$u = (1,0,0,0)\ ; \qquad u^2 = g_{\mu\nu}u^\mu u^\nu = -1\ . \tag{20.3}$$

In these expressions, we neglected all effects that are of higher order in the particle's mass μ, since μ is chosen to be small.

Written this way, we can now easily give this particle a Lorentz boost. In the boosted frame we can take

$$mu^\mu = p^\mu \to (p,0,0,p)\ , \qquad Gp = \frac{\mu v}{\sqrt{1-v^2/c^2}} \gg \mu\ . \tag{20.4}$$

In the limit $\mu \to 0$, p fixed, one has $r \to |x\cdot u|$.

It will turn out to be useful to compare this metric with the flat space-time metric in two coordinate frames $y^\mu_\pm$, defined as

$$y^\mu_{(\pm)} = x^\mu \pm 2\mu\, u^\mu \log r\ . \tag{20.5}$$

We have

$$\mathrm{d}y^2_{(\pm)} = \mathrm{d}x^2 \pm \frac{4\mu}{r}(u\cdot\mathrm{d}x)\mathrm{d}r - 4\mu^2\frac{\mathrm{d}r^2}{r^2}\quad ; \tag{20.6}$$

$$\mathrm{d}s^2 - \mathrm{d}y^2_{(\pm)} = \frac{2\mu}{r}\mathrm{d}\Big(r \mp (u\cdot x)\Big)^2 + 4\mu^2(\mathrm{d}\log r)^2\quad . \tag{20.7}$$

Now consider the limit (20.4). We keep p fixed but let μ tend to zero. We now claim that when $(p\cdot x)>0$, the metric $\mathrm{d}s^2$ approaches the flat metric $\mathrm{d}y^2_{(+)}$, whereas when $(p\cdot x)<0$, we have $\mathrm{d}s^2\to\mathrm{d}y^2_{(-)}$, and at the plane defined by $(p\cdot x)=0$ these two flat space-times are glued together according to

$$y^\mu_{(+)} = y^\mu_{(-)} + 4p^\mu \log|\tilde x|\ , \tag{20.8}$$

where $\tilde x = (0,x,y,0)$ are the transverse part of the coordinates y^μ.

This is seen as follows. First, we note that the last term of Eq. (20.7) can be ignored. Next, given a small positive number λ, we divide space-time in three regions:

$$\begin{array}{lrcl} A) & (u\cdot x) & > & \lambda\ ; \\ B) & (u\cdot x) & < & -\lambda\ , \\ C) & |(u\cdot x)| & \le & \lambda\ . \end{array} \tag{20.9}$$

In region (A), we use

$$r-(u\cdot x) = \frac{x^2}{r+(u\cdot x)}\ , \tag{20.10}$$

which is therefore bounded by $\frac{x^2}{\lambda}$. Thus, the first term in Eq. (20.7) for $y_{(+)}$ is bounded by $\frac{\mu}{\lambda^2}$ times a coordinate dependent function (note that $r\ge|\tilde x|$). Similarly, in region (B), Eq. (20.7) for $y_{(-)}$ will tend to zero as μ/λ^2. In the region (C), we have that r

and $(u \cdot x)$ are both bounded by terms that are finite or proportional to λ. So, in (C), both equations (20.7) are bounded by functions of the form μ or $\mu\lambda^2$. Choosing λ such that, as $\mu \to 0$, both $\mu\lambda^2 \to 0$ and $\mu/\lambda^2 \to 0$, allows us to conclude that

$$\begin{array}{llll} A) & \mathrm{d}s^2 \to \mathrm{d}y_{(+)}^2 & \text{if} & (p\cdot x) \geq 0 \ , \\ B) & \mathrm{d}s^2 \to \mathrm{d}y_{(-)}^2 & \text{if} & (p\cdot x) \leq 0 \ , \\ C) & y_{(+)} = y_{(-)} + 4\mu\, u^\mu \log r & \text{at} & (p\cdot x) \approx 0 \ , \end{array} \tag{20.11}$$

which is equivalent to Eq. (20.8). This defines the Aechelburg-Sexl metric.

Defining $x^\pm = z \pm t$, one finds for a source particle moving with the speed of light to the positive z-direction that two flat space-times, one with coordinates $(x^\pm_{(+)},\, \tilde x_{(+)})$ and one with coordinates $(x^\pm_{(-)},\, \tilde x_{(-)})$ are connected together at the point $x^-_{(+)} = x^-_{(-)}$, in such a way that $\tilde x_{(+)} = \tilde x_{(-)}$ and

$$x^+_{(+)} - x^+_{(-)} = 4Gp^+ \log|\tilde x| = 8Gp\log|\tilde x| \ . \tag{20.12}$$

The r.h.s. of this equation happens to be a Green function,

$$\delta x^+ = -pf(\tilde x) \ , \qquad \tilde\partial^2 f(\tilde x) = -16\pi G\delta^2(\tilde x) \ . \tag{20.13}$$

This result can be generalized to describe a light particle falling into the horizon of a black hole. For the Schwarzschild observer, its energy is taken to be so small that its gravitational field appears to be negligible, and the black hole mass will hardly be affected by the energy added to it. However, in Kruskal coordinate space, see Eqs. (6.3)–(6.7), the energy is seen to grow exponentially as Schwarzschild time t progresses. Let us therefore choose the Kruskal coordinate frame such that the particle came in at large negative time t. This means that in Eq. (6.3), $x \approx 0$, or, the particle moves in along the past horizon. In view of the result derived above, one can guess in which way the particle that goes in will deform the metric: we cut Kruskal space in halves across the x-axis, and glue the pieces together, again after a shift, defined by

$$y_{(+)} = y_{(-)} - 16\pi G\, p_x F(y_{(-)}, \theta, \varphi) \ , \tag{20.14}$$

where $(\pm)$ now refers to the regions $x > 0$ and $x < 0$. This corresponds to a metric with a delta-distributed Riemann curvature on the plane $x = 0$. The function F is yet to be determined.

By demanding that the *Ricci* curvature must still vanish at the seam, one can compute the equations for F. It is then found that F has to obey

$$-\tilde\partial^2 F + F = \tilde\delta^2(\Omega) \ , \tag{20.15}$$

where $\tilde\partial^2$ is the spherical Laplacian $\ell(\ell+1)$ and $\tilde\delta^2(\Omega)$ the Dirac delta function on the sphere $(\theta,\, \varphi)$. The quantity p_x is the momentum of the particle falling in, with respect to the Kruskal coordinate frame. This equation, which clearly shows a strong similarity with the case derived earlier for Rindler space, can be solved in an integral form. It turns out not to depend on $y_{(-)}$ itself. At small angular distances θ, one gets

$$F(\theta) \to (1/2\pi)\log(1/\theta) \ . \tag{20.16}$$

It is important to note that this shift in the Kruskal y coordinate affects the Hawking radiation. It does *not* affect its thermal nature, not the temperature itself, but it will affect the microstates. This may be an important starting point for further investigations of the quantum structure of a black hole.

21. History

A brief history of black holes in General Relativity:[8]

- **1915**: Einstein formulates the general theory of relativity.
- **1916**: Karl Schwarzschild publishes his exact spherically symmetric and static solution, showing a singularity at $r = 2M$.
- **1924**: Eddington introduces coordinates that are well behaved at $r = 2M$.
- **1930**: Using general relativity, Subrahmanyan Chandrasekhar calculates that a non-rotating body of electron-degenerate matter above 1.44 solar masses (the Chandrasekhar limit) would collapse.
- **1933**: LeMaître realizes the significance of Eddington's result: $r = 2M$ is a fictitious singularity.
- **1958**: David Finkelstein introduces the concept of the event horizon by presenting Eddington-Finkelstein coordinates, which enabled him to show that "The Schwarzschild surface $R = 2M$ is not a singularity, but that it acts as a perfect unidirectional membrane: causal influences can cross it in only one direction". All theories up to this point, including Finkelstein's, covered only non-rotating black holes.
- **1960**: Kruskal and Szekeres obtain the maximal extension of the Schwarzschild solution.
- **1960**: Penrose introduces global methods in the study of General Relativity.
- **1963**: Roy Kerr finds a generalization of the Schwarzschild metric and interprets it as the field of a "spinning particle".
- **1967**: John Wheeler uses the words "black hole" in a public lecture. Unofficially, the phrase has been used earlier by others.
  Black hole uniqueness theorems make people believe that black holes cannot form, because time reversal invariance of Nature's laws would then imply that only perfectly symmetric initial states could collapse gravitationally. Roger Penrose saw the flaw of that argument: there may be perturbations in the black hole metric in the form of multipole components, but they all die out or radiate away exponentially.

[8]I made use here of notes made by A. Ashtekar

- **late 1960's - early 1970's**: Bekenstein, Bardeen, Carter, Penrose and Hawking explore the structure and properties of black holes. Bekenstein proposes that black holes should carry entropy, proportional to the horizon area. Bardeen, Carter and Hawking prove the first theorems on black hole mechanics.

- **1974**: Hawking discovers black hole evaporation. Quantum fields on a black hole background space-time radiate thermal (*i.e.* black body) spectrum of particles, with a temperature of $kT = \hbar\kappa/2\pi$.

- **1982**: Bunting and Mazur independently derive a generalized uniqueness theorem: any isolated, time-independent black hole in general relativity is described by the Kerr metric. hence the equilibrium state of every (uncharged) black hole is fully described by only two parameters: mass and angular momentum (represented by M and J).

- **1995**: Strominger, Vafa, Maldacena and others discover how to describe the black hole microstates in terms of D-branes in string theory. The description is particularly detailed at or near the extreme limit, and usually the black hole is considered in more than 4 dimensional space-time.